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# STRENGTH AND ELASTICITY OF STRUCTURAL MEMBERS

BY

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GENERAL

## PREFACE.

THESE chapters were originally written as a series of lectures for students at the Royal Indian Engineering College, Cooper's Hill; so that the book may be looked upon as one mainly for students of Engineering.

The aim has been to make the work as practical as possible; and to keep the methods simple and concise, involving only a fair knowledge of elementary mathematics. Numerous diagrams and illustrations have been introduced, so as to enable the student to obtain as clear an insight into the methods as possible.

Many of the proofs are of course similar to those ordinarily used in other published works, and I have especially to acknowledge the following books of reference:

<i>Strength and Elasticity of Materials.</i>	Professor EWING.
<i>Theory of Structures and Strength of Materials.</i>	Professor BOVEY.
<i>Elements of Machine Design.</i>	Professor UNWIN.
<i>Notes on Engineering Construction.</i>	Professor REILLY.

I have to thank Professor Minchin, F.R.S., for much advice and assistance, and I am also indebted to Dr Brightmore, D.Sc., for his help.

A large number of Examples have been added as exercises for the student on the application of the principles explained in each chapter.

R. J. W.

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## CHAPTER I.

### GRAPHIC STATICS.

1. In order that a line may represent a force, it is necessary, (a) that its length to a given scale should represent the magnitude of the force, (b) its direction must correspond with the line of action of the force, (c) the *sense* of the force must be indicated by an arrow.

#### 2. Triangle of forces.

Let  $OA$  and  $OB$  (Fig. 1) represent in magnitude and direction two forces  $P$  and  $Q$  acting at a point  $O$ , then  $R$  the diagonal of the parallelogram represents the resultant of  $P$  and  $Q$  in magnitude and

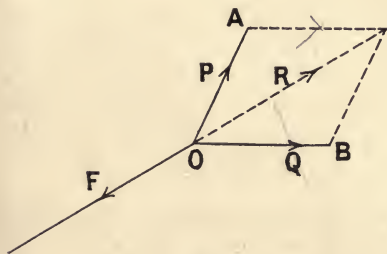


Fig. 1.

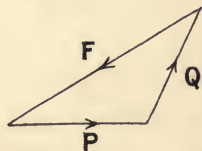


Fig. 2.

direction ; and the force  $F$  which, acting with  $P$  and  $Q$  will keep the point  $O$  in equilibrium, must be equal and opposite to  $R$ , and in the same straight line.

Thus, when three forces act at a point, they will be in equilibrium if they are parallel and proportional to the sides of a triangle which are marked *with arrows all going round the triangle in the same sense* (Fig. 2).

### 3. Resultant of a number of forces acting at a point.

Let  $P_1, P_2, P_3, P_4$  be the forces acting at  $O$  (Fig. 3). Draw  $ab$  (Fig. 4) parallel and proportional to  $P_1$ , and from the extremity  $b$

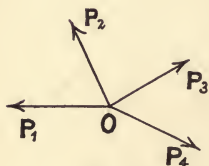


Fig. 3.

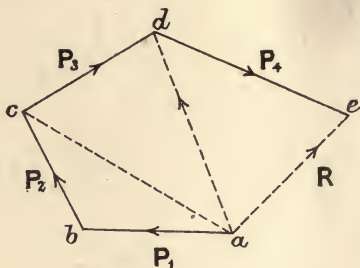


Fig. 4.

draw  $bc$  parallel and proportional to  $P_2$ , the resultant of  $P_1$  and  $P_2$  is represented by  $ac$ . Then draw  $cd$  parallel and proportional to  $P_3$ , the resultant of  $ac$  and  $P_3$  is  $ad$ . Finally drawing  $de$  parallel and proportional to  $P_4$  and compounding it with  $ad$ , we get the resultant of all the forces represented in magnitude and direction by  $ae$ , the closing side of the polygon. Thus, if a polygon be drawn with its sides successively parallel and proportional to the forces acting at the point, then the side which is required to close the polygon, represents the resultant of all the forces in magnitude and direction. In order that the system of forces may be *in equilibrium* the resultant force must be zero; that is, the *polygon of forces must close*.

Hence, if any number of forces in one plane acting at a point are in equilibrium, then lines drawn successively parallel to the forces, in the same direction as the sense of the forces, and having lengths proportional to the magnitudes of the forces, must form a closed polygon—or, in short, the polygon of forces must close, the direction arrows all pointing round the polygon *in the same sense*.

### 4. Forces acting in one plane which do not meet in a point. Conditions of Equilibrium. Funicular Polygon.

If three forces are in equilibrium, their directions must pass through a point, and the condition of equilibrium is that stated in Art. 3.

Let  $P_1, P_2, P_3, P_4$  (Fig. 5) be a system of forces acting in one plane on a body. From any point  $a$  (Fig. 6) draw lines  $ab, bc, cd, de$  successively parallel and proportional to the given forces. The figure  $abcde$  is the force polygon of the system. The closing line  $ae$  represents the magnitude and direction of the resultant, its sense being opposite to that of the other forces followed in circuit round the polygon. Take any pole  $o$ , and from it draw lines  $oa, ob, oc, od, oe$ , to the

vertices of the force polygon. From any point  $f$  on the line of action of  $P_1$  draw  $fg$  connecting  $P_1$  and  $P_2$  parallel to the ray  $ob$ , which comes between  $ab$  and  $bc$  representing  $P_1$  and  $P_2$ ; also from  $f$  draw  $fm$  parallel to the ray  $oa$  which is between  $P_1$  and  $R$ . From  $g$  the point

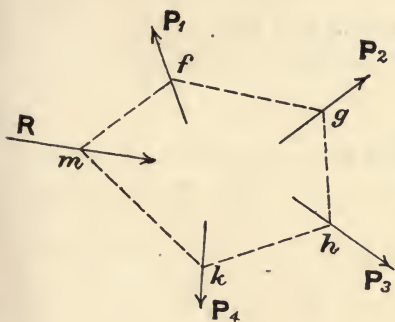


Fig. 5.

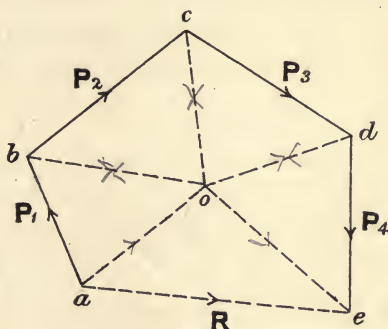


Fig. 6.

where  $fg$  meets  $P_2$  draw  $gh$  parallel to  $oc$ , meeting  $P_3$  in  $h$ ; from  $h$  draw  $hk$  parallel to  $od$ , meeting  $P_4$  in  $k$ ; and lastly draw  $km$  parallel to  $oe$  meeting  $fm$  in  $m$ . Then  $m$  is a point on the resultant  $R$ ; through this point draw a line parallel to  $ae$ , and we have the resultant fixed in magnitude, position, and direction.

The lines  $fg$ ,  $gh$ ,  $hk$ ,  $km$ ,  $mf$ , parallel to the rays drawn from any pole  $o$  to the vertices of the force polygon, form what is called the *funicular polygon* or *link polygon* of the system of forces.

If in Fig. 5 we apply a force  $P_5$  at  $m$ , equal and opposite to  $R$ , then the forces  $P_1 \dots P_5$  will be in equilibrium. The resultant must always pass through the point of intersection of the extreme sides  $fm$  and  $km$  of the funicular. For each force can be resolved into its components along the two sides of the funicular polygon which meet at the vertex where the force acts, these components will neutralize each other, with the exception of those in the lines  $km$  and  $fm$ . Thus the whole system is reduced to two forces in these lines, which are represented in magnitude by  $oe$  and  $oa$  the components of  $R$ .

Since the pole  $o$  may be taken at any convenient point, either inside or outside the force polygon, and as the point  $f$  where the two components of  $P_1$  are assumed to act, may be taken anywhere on the line of action of  $P_1$ , we see that any number of funicular polygons may be drawn for the system of forces.

### 5. Graphic conditions of Equilibrium.

As explained in Art. 4 the resultant of a system of forces in one plane acting on a body is proportional to the line required to close the force polygon, now for equilibrium this line must be zero; therefore

the force polygon must close of itself. Again, the given system is equivalent to forces represented in magnitude and direction by  $ao$  and  $oe$ , their lines of action being in the first and last sides of any funicular; these lines must coincide, that is, the funicular polygon must close. Thus, the conditions of equilibrium are:

- (1) *The force polygon of the system must close.*
- (2) *Any funicular polygon of the system must close.*

## 6. Stress.

When a body is acted on by external forces which tend to deform it, the force exerted in the interior of the body, which resists deformation, is called *stress*.

Consider a bar  $AB$  which is being pulled (Fig. 7), or pushed (Fig. 8), by two equal and opposite forces  $P$ . Conceive a section dividing the bar into two portions  $C$  and  $D$ . Considering the equilibrium of  $C$ , it is seen that the force  $P$  to the left is balanced by the force which  $D$  exerts on  $C$  at the section. Similarly for the equilibrium of  $D$ , the force exerted on  $D$  by  $C$  at the section must be equal and opposite to  $P$ . Thus on opposite sides of any ideal section there exist two equal and opposite forces, each equal to  $P$ .

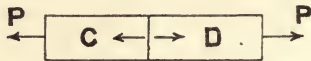


Fig. 7.

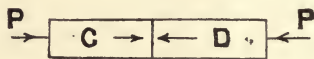


Fig. 8.

This action is called *stress*. The effect of the load is to produce stress on the section, and the *total stress* on the section is equal to  $P$ .

The nature of the internal stress depends on the external forces. If the forces tend to pull  $C$  and  $D$  apart, the stress is *tensile*; if the forces push  $C$  and  $D$  together the stress is *compressive*. *Ties* are members in tension. *Struts* are members in compression.

## 7. Stresses in a closed articulated polygonal frame.

A frame is a structure made up of straight rigid bars articulated or hinged at the ends. The following conditions are assumed:

- (1) The hinge-pins are without friction.
- (2) All external forces acting on the frame are applied at and on the joints.

Thus in the case of a frictionless pin-joint, the external force is taken as acting through the centre of the pin, and the stress exerted on the pin by a bar, which is equal and opposite to the reaction on the bar by the pin, acts along the normal to the surface of contact; and the stresses in the bars must have their lines of action coincident with the straight lines joining the centres of the pins.

If  $P$  and  $Q$  (Fig. 9) are two forces acting at the joints  $A$  and  $B$  respectively of a jointed frame,  $P$  can be resolved along the bars  $CA$  and  $AB$ ; and  $Q$  along the bars  $AB$  and  $BD$ . These components are

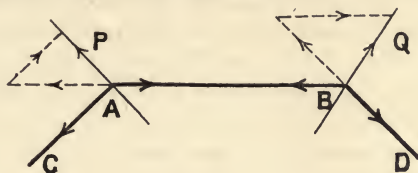


Fig. 9.

shown by dotted lines; the stresses in the bars will be equal and opposite to the resolved components of the external forces  $P$  and  $Q$ .

The stresses in the two bars meeting at a joint, together with the external force acting at that joint, must form a system of forces in equilibrium.

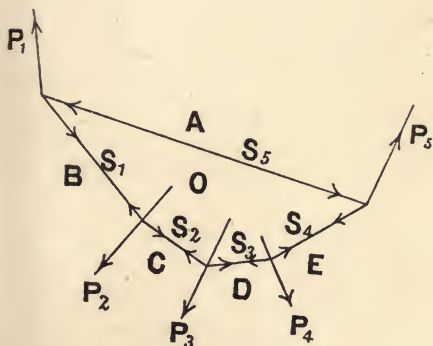


Fig. 10.

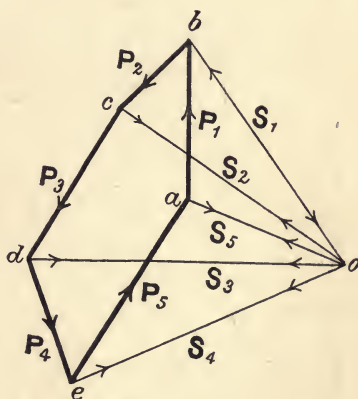


Fig. 11.

*Notation.* For finding the stresses in a frame the most convenient method of lettering is that of Henrici and Bow, in which letters are placed in the closed spaces between members, and in open spaces separated by the lines of action of the external forces. Each member and force is then designated by the letters of the spaces which it separates. The joints are named by the letters round them.

Let Fig. 10 be a polygonal frame lettered in this manner; and in equilibrium under the action of the external forces  $P_1, P_2, P_3, P_4, P_5$ . Required to find the stresses  $S_1$  ( $ob$ ) in  $OB$ ,  $S_2$  ( $oc$ ) in  $OC$ ,  $S_3$  ( $od$ ) in  $OD$ ... in the bars of the frame.

As the external forces are in equilibrium, the force polygon must close. Draw the force polygon  $abcde$  (Fig. 11)  $ab$  being parallel and proportional to  $AB$ ;  $bc$  parallel and proportional to  $BC$ ... Now the

joint  $OBC$  is in equilibrium under the action of  $P_2$ , and the stresses  $S_1$  and  $S_2$  in  $OB$  and  $OC$  respectively. Draw from  $b$  and  $c$  lines  $bo$  and  $co$  parallel to  $BO$  and  $CO$ , these intersect in  $o$ , giving  $ob$  as the stress  $S_1$  in  $OB$  and  $co$  as the stress  $S_2$  in  $CO$ . Similarly  $P_3, S_2, S_3$  form a triangle  $cod$  which has one side  $co$  common to the triangle  $cb o$ . Thus at each apex the external force may be resolved into components in the two given directions, and the stresses  $S_1, S_2, S_3, S_4, S_5$  found. To find the *sense* of the stresses it is only necessary to follow round in circuit the sides of the force triangle *for each apex separately*, starting with the direction of the *given* force, and then transfer these directions to the corresponding apex of the frame. Thus at joint  $OBC$ , the direction of  $P_2$  is given, it acts in Fig. 11 from  $b$  to  $c$ , therefore following round the triangle  $bco$ , the stress  $S_2$  acts from  $c$  to  $o$ , and the stress  $S_1$  from  $o$  to  $b$ ; these arrows being transferred to the frame diagram (Fig. 10), we see that  $S_1$  and  $S_2$  both act *away from* the joint  $BOC$ , that is both are *tensile*. Again for the joint  $COD$ , we get from the force triangle  $cdo$ , the stress  $S_3$  in the direction  $do$ , and the stress  $S_2$  in the direction  $oc$ . Similarly for the stresses in the other bars. It will be found that the stress  $S_5$  ( $oa$ ) is *compressive*, the arrows pointing *towards* the joints.

Fig. 10 is called the frame diagram, and Fig. 11 the force or stress diagram. To every side of one figure there is *one* corresponding side in the other figure; these corresponding sides are parallel to each other; and each group of lines meeting in a point of one figure form a closed polygon in the other. The two figures are therefore described as "*reciprocal*" to each other.

Suppose that in Fig. 10, the magnitudes of  $P_1$  and  $P_5$  are unknown, their lines of action being given. Beginning at  $b$  (Fig. 11) draw  $bcde$  the polygon for the known forces  $P_2, P_3, P_4$ ; then lines drawn from  $b$  and  $e$  parallel to  $P_1$  and  $P_5$  intersect in  $a$  giving  $ab$  and  $ea$  the magnitudes, directions, and sense of these forces. Draw a funicular polygon with respect to any pole  $o$ , and produce the extreme sides  $OB$  and  $OE$  to meet the lines of action of  $P_1$  and  $P_5$ . Draw the closing line  $AO$ ; then  $OA$  will be parallel to  $oa$ . This property is important when we wish to find the reactions of supports in the case of parallel forces. *For in the case of parallel forces*, the polygon of forces is a straight line, consequently  $ba$  and  $ea$  would not intersect; hence in order to determine  $a$  we must draw a ray  $oa$  parallel to  $OA$ , the closing line of the funicular polygon.

## 8. Dead loads. Reactions, Shearing forces, and Bending Moments.

The *shearing force* at any section of a beam is the algebraic sum of the external forces between the section and either end of the beam. The *bending moment* at a section is the algebraic sum of the moments

of the external forces, acting between the section and either support, about an axis in the section.

Let Fig. 12 represent a horizontal girder supported at the ends, and in equilibrium under the action of the fixed vertical loads  $W_1$ ,  $W_2$ ,  $W_3$  and the two upward reactions  $R_1$  and  $R_2$ .

Construct the force polygon (Fig. 15) by drawing the line of loads  $ad$  to represent the sums of the loads on a suitable scale, and divide  $ad$  into segments  $ab$ ,  $bc$ ,  $cd$  representing respectively the given loads  $W_1$ ,  $W_2$ ,  $W_3$ . Choose a convenient pole  $o$ , such that the polar distance  $oh = H$  represents on the same scale an even number of units of force. Complete the force diagram and draw the corresponding funicular (Fig. 13), its sides intersecting on the vertical lines of action of the loads.

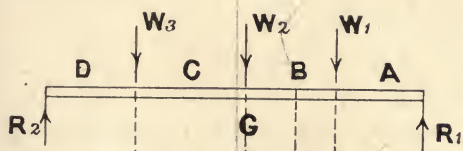


Fig. 12.

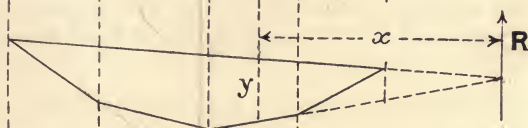


Fig. 13.

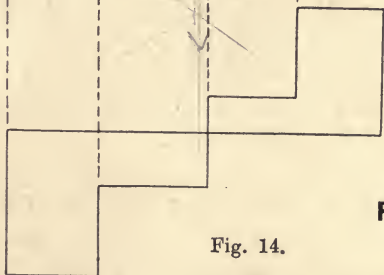


Fig. 14.

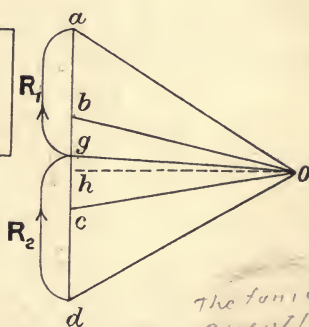


Fig. 15.

**Reactions.** Complete the funicular by drawing the closing line between  $R_1$  and  $R_2$ , and draw  $og$  parallel to it in the force diagram, thus determining the point  $g$ ; then  $dg$  and  $ga$  are the two reactions  $R_2$  and  $R_1$  respectively, which acting upwards close the force polygon.

**Shearing force** (Fig. 14). Between the right support and the first load  $W_1$ , the vertical shear is equal to the reaction  $ag$ ; between  $W_1$  and  $W_2$  the vertical shear is  $ga - ab = gb$ ; between  $W_2$  and  $W_3$  it is  $ga - ab - bc = gc$ ; between  $W_3$  and left support it is  $gd$ . The ordinates of the stepped figure measured from the line drawn through

*The funicular is  
presently drawn  
the mirror image  
fig. 1*

$g$  perpendicular to the load line  $ag$ , give the shearing force throughout the span to the same scale as that used for the line of loads.

*Bending moment* (Fig. 13). Let the bending moment  $M$  be required at any section of the beam, say, between  $W_1$  and  $W_2$ . Draw a vertical through the section cutting two sides of the funicular, and let the ordinate intercepted between them be called  $y$ . These sides if produced give the point of application of the resultant of  $R_1$  and  $W_1$ , the magnitude of which is represented by  $bg$  in the force polygon; let this resultant be  $R$ , and its horizontal distance from  $y = x$ . Then

$$M = Rx.$$

But since  $ho = H$  is the altitude of the triangle  $bgo$  on base  $bg$  and

$x$     "    "    "    triangle on base  $y$ ,

we have 
$$\frac{R}{H} = \frac{y}{x} \text{ or } Rx = Hy,$$

therefore the bending moment on the section is

$$M = Hy,$$

and since  $H$  is constant, *the bending moment at every point in the span is proportional to the vertical ordinate of the funicular polygon at that point*. If the force represented by  $H$  is called unit of force, then

$$M = y$$

and the value of  $H$  determines the scale of bending moments on which  $y$  represents  $M$ .

For example, if the scale of lengths be  $\frac{1}{4}$  inch = 1 foot, and  $H = 20$  tons on the scale of forces, then  $H \times \frac{1}{4}$  inch = 20 ft.-tons, or,

$$1 \text{ foot-ton} = \frac{1}{20} \times \frac{1}{4} = \frac{1}{80} \text{ inch};$$

that is each  $\frac{1}{80}$  of an inch on a vertical ordinate of the funicular polygon represents 1 foot-ton of bending moment.

## 9. Graphic construction for the centre of area of plane figures.

*Triangle*. The centre of area is at the intersection of lines drawn from any two angles bisecting the opposite sides.

*Parallelogram*. The centre of area is at the intersection of the two diagonals.

*Any Plane Quadrilateral*. Let  $ABCD$  (Fig. 16) be any quadrilateral figure; draw the diagonals  $AC$  and  $BD$ . Measure  $DE = BF$ , and  $CG = AF$ , then the centre of area  $O$  of the whole figure coincides with the centre of area of the triangle  $FEG$ .

*Centre of area of any two plane surfaces whose respective centres of area are already known*.

Let  $C_1$  and  $C_2$  (Fig. 17) be the known centres of area of the two surfaces whose areas are  $A_1$  and  $A_2$  respectively. Join  $C_1$  and  $C_2$ , from

$C_2$  set off a line  $C_2B_2$  in any direction whose length represents on a given scale the area  $A_1$ ; and from  $C_1$  set off a line  $C_1B_1$  on the

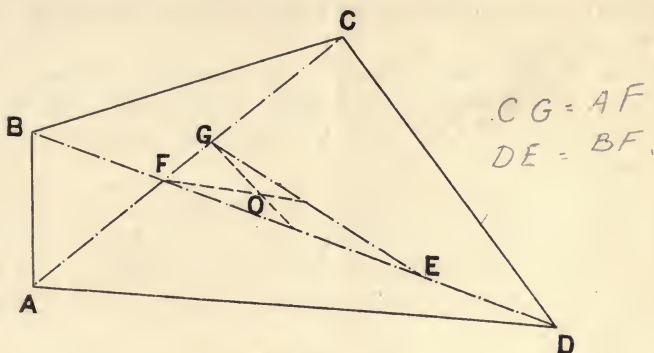


Fig. 16.

opposite side of  $C_1C_2$ , parallel to  $C_2B_2$ , whose length on the same scale represents the area  $A_2$ . Join  $B_1B_2$ , the intersection of  $B_1B_2$  and  $C_1C_2$  is the centre of area  $O$ .

A repetition of this process applied to  $O$ , and the known centre of area of a third surface, will give the common centre of area of the three.

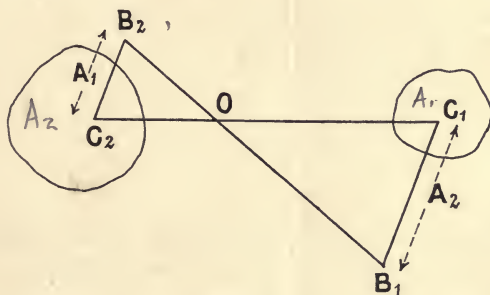


Fig. 17.

*Polygons.* The centre of area of a polygon can be found by dividing it up into triangles or four-sided figures, and finding the centre of area of each. At the known centres of areas suppose vertical forces to act proportional to the areas. Draw the force diagram and funicular, and find the line of action of the resultant  $R$ ; the common centre of area must lie somewhere in this line. Again, draw the lines of action of the parallel forces through the centres of area in any other direction (say horizontal). Construct a new force polygon and a new funicular, and find the line of action of the resultant. The intersection of this latter resultant with the former one gives the common centre of area.

## 10. Roof trusses.

A *roof truss* consists of a frame supporting a covering. Roof trusses are sometimes called *principals*, and are placed about 10 or 12 feet apart.

The simplest form of frame is a *triangle*; its shape cannot be changed without altering the lengths of the sides. For this reason, all complex trusses for roofs or bridges are made up of triangular frames.

**LOADS ON ROOFS.** The loads on a roof truss are (*a*) the dead load due to the weight of the framing, covering and snow, (*b*) the live load due to wind. The weight of the framing depends on the type and design of the roof, and may vary from 5 to 20 lbs. per square foot of area covered.

The following table gives the weight of various roof coverings in lbs. per square foot.

Description	Weight in lbs. per square foot
Slates	5 to 11
Tiles	7 to 20
Timbering for tiled and slated roofs (additional)	$5\frac{1}{2}$ to $6\frac{1}{2}$
Corrugated sheets	$3\frac{1}{2}$
Boarding $\frac{3}{4}$ inch	$2\frac{1}{2}$
Boarding and sheet-iron (20 gauge)	$6\frac{1}{2}$

The snow load is usually taken in England as 5 or 6 lbs. per square foot. In colder latitudes this would have to be doubled.

**WIND PRESSURE.** In England the maximum pressure of wind is usually taken to vary between 40 and 50 lbs. per square foot of surface perpendicular to its direction, which may be taken as horizontal.

The pressure on an *inclined surface* can be got from the following empirical formula deduced experimentally by Hutton,

$$p_n = p \sin \theta^{1.84 \cos \theta - 1},$$

where  $p$  is the intensity of the horizontal wind pressure on a vertical surface, and  $p_n$  the normal intensity on a surface inclined at an angle  $\theta$  to the wind's direction. For the inclined surface of a roof, if the wind is taken as horizontal,  $\theta$  is the pitch of the roof.

The following table gives the value of  $p_n$  for  $p$  equal to 40 lbs. per square foot, and for roofs of different pitch.

Pitch of roof	$p_n$
5°	5.0
10°	9.7
15°	14.2
20°	18.1
25°	22.6
30°	26.4
35°	30.0
40°	33.3
45°	36.0
50°	38.1
55°	39.4
60°	40.0

For other values of  $p$ , the corresponding values of  $p_n$  will be directly proportional to  $p$ .

The following formula is often used for determining  $p_n$  the intensity of the normal pressure on a surface inclined at an angle  $\theta$  to the direction of the wind,

$$p_n = p \frac{2 \sin \theta}{1 + \sin^2 \theta},$$

where  $p$  is the intensity of the wind pressure on a surface perpendicular to its direction.

If  $v$  be the velocity of the wind in feet per second,

$$p = \left(\frac{v}{20}\right)^2 \text{ approx.}$$

*When  $v = 40 \frac{\text{ft}}{\text{sec}}$ , the corresponding  $V = 20 \sqrt{40} \frac{\text{ft}}{\text{sec}} = 86 \frac{\text{miles}}{\text{hour}}$*

**DISTRIBUTION OF LOAD. REACTIONS.** The dead load and wind load are usually taken as uniformly distributed. The rafters are generally assumed to be divided up at the joints into short supported beams, so that each joint carries *one half* the load between the two adjacent supports.

In roofs of small span, the two ends of the roof truss are *fixed*; and with vertical loading, the two reactions are vertical; but for a wind load, which is assumed to act on one side of the roof only, both reactions due to it are inclined, and parallel to the normal wind pressure.

For roofs of large span, one end of the truss is bolted down, or fixed to the support; and the other end is supported on expansion rollers. In this case the reaction due to the wind at the *fixed end* will be inclined, and that at the *free end* will be vertical. The total horizontal component of the wind pressure must be carried at the fixed end.

FRAME AND STRESS DIAGRAMS. NOTATION. The drawing of the framework is called the *frame diagram*, and the reciprocal figure the *stress diagram*.

The sides of the stress diagram are proportional to the external forces, and to the stresses in the corresponding bars of the frame. Having drawn lines on the frame diagram to represent the external forces at the joints, a letter is given to each enclosed area of the frame, also to each open space between the lines of action of the external forces. Each bar of the frame is designated by the two letters in the spaces separated by the bar. The line parallel to it in the stress diagram is similarly lettered at its extremities.

### EXAMPLE I. A KING ROOF TRUSS.

*Data.* Span = 30 feet.

Distance between principals = 10 feet.

Pitch =  $30^\circ$ .

Dead load on roof per square foot of horizontal surface covered = 15 lbs.

Horizontal wind pressure per square foot of vertical surface = 40 lbs.

In this first example we will take the vertical load and the wind load separately.

*Vertical load.* Fig. 18 is the frame diagram, and Fig. 19 the stress diagram. The fixed vertical load on one truss is

$$10 \times 30 \times 15 = 4500 \text{ lbs.} = 2 \text{ tons (app.)}$$

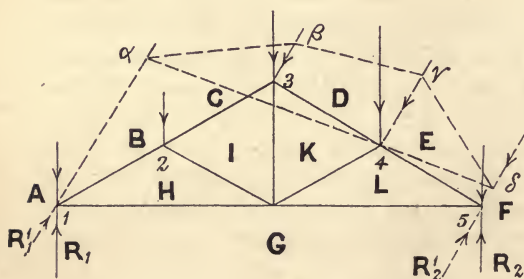


Fig. 18.

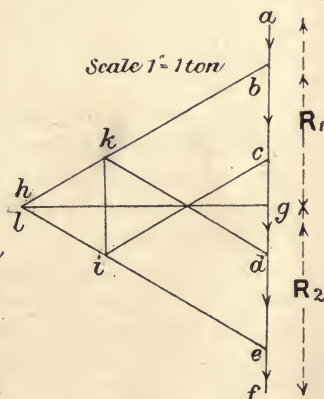


Fig. 19.

distributed as follows. Load at each of the joints 1 and 5 =  $\frac{1}{4}$  ton. Load at each of the joints 2, 3, 4 =  $\frac{1}{2}$  ton. The loads being vertical and symmetrical, the reactions will be vertical, and each reaction will be equal to half the total load, that is 1 ton.

*Stress diagram.* To draw the stress diagram, the reactions of the supports must *first* be found, and then the polygon of external forces, including the reactions, must be drawn; this polygon must close since the system of external forces is in equilibrium. Commencing at *a* draw the vertical line of loads *af* (Fig. 19), set off to any suitable scale  $ab = AB = \frac{1}{4}$  ton,  $bc = BC = \frac{1}{2}$  ton,  $cd = CD = \frac{1}{2}$  ton,  $de = DE = \frac{1}{2}$  ton,  $ef = EF = \frac{1}{4}$  ton. Bisect *af* in *g*, then *fg* = *ga* represent the equal reactions  $R_1$  and  $R_2$ .

Again, the external load, acting at any one joint of the frame, must be in equilibrium with the stresses in all the bars meeting at that joint.

Consider the equilibrium of joint 1. From *b* (Fig. 19) draw *bh* and *gh* respectively parallel to *BH* and *GH* (Fig. 18), then the figure *abhga* is the stress diagram for joint 1, and *bh* and *hg* give the stresses in *BH* and *HG*.

To determine the *sense* of the stresses in the bars *BH* and *HG*. Since the joint 1 is in equilibrium, the arrows must point the same way round *abhga*. The direction arrow of *ab* is known, and this fixes the other arrows. Transferring these arrows to the corresponding bars of the frame diagram, we see that the arrow of *BH* points *towards* the joint 1, and that of *GH* *away* from it.

Hence *BH* is in compression and *GH* in tension.

Next, consider the equilibrium of joint 2. *hb* and *bc* are known — *ci* and *ih* are unknown. Draw *ci* and *hi* respectively parallel to *CI* and *HI*. Then the figure *bcihb* is the reciprocal of joint 2, and *ci* and *hi* give the stresses in bars *CI* and *HI*, and following the arrows given by the known direction of *bc*, we see that bars *CI* and *HI* are both in compression.

Similarly for joint 3, we get the stress figure *cdkic*, the stress in bar *DK* = *dk* being compressive, and that in bar *KI* = *ki* being tensile.

If at any joint there is no external load, the direction of the arrows passing round the corresponding polygon must be got from the character of the stress already found in one of the members at the joint.

The complete stress diagram (Fig. 19), is thus obtained by considering each joint in succession, taking care to start from, and then proceed to, a joint at which there are only *two unknown* forces.

The stresses got by scale are :

In bar	<i>BH</i> , <i>bh</i> = 1·5	tons, compressive
„	<i>EL</i> , <i>el</i> = 1·5	„ „
„	<i>CI</i> , <i>ci</i> = 1·0	„ „
„	<i>KD</i> , <i>kd</i> = 1·0	„ „
„	<i>HI</i> , <i>hi</i> = 0·5	„ „
„	<i>KL</i> , <i>kl</i> = 0·5	„ „
„	<i>IK</i> , <i>ik</i> = 0·5	„ tensile
„	<i>HG</i> , <i>hg</i> = 1·30	„ „
„	<i>LG</i> , <i>lg</i> = 1·30	„ „

**WIND PRESSURE.** The surface of the roof being inclined at  $30^\circ$  with the horizontal, and the intensity of the wind pressure on a surface normal to its horizontal direction being given as 40 lbs. per square foot, the intensity of the pressure normal to the surface of the roof is from table Art. 10

$$p_n = 26.4 \text{ lbs. per square foot of roof surface.}$$

This normal pressure acts on one side only of the roof, and the total pressure on one bay 10 feet long is

$$10 \times 15 \times \sec 30^\circ \times 26.4 \text{ lbs.} = 4573 \text{ lbs.} = 2.04 \text{ tons.}$$

The load at joint 4 is 1.02 tons, that at joints 3 and 5 being each equal to 0.51 tons.

**Reactions.** Assume the truss *fixed* at both ends. As the sum of the external forces in the direction of the wind loads must be zero, the reactions are parallel to them and can be found by constructing the funicular polygon, thus:

In Fig. 20 draw the inclined line of loads  $cf$ , and set off on it in order

$$cd = CD = 0.51 \text{ ton,}$$

$$de = DE = 1.02 \text{ tons,}$$

$$ef = EF = 0.51 \text{ ton.}$$

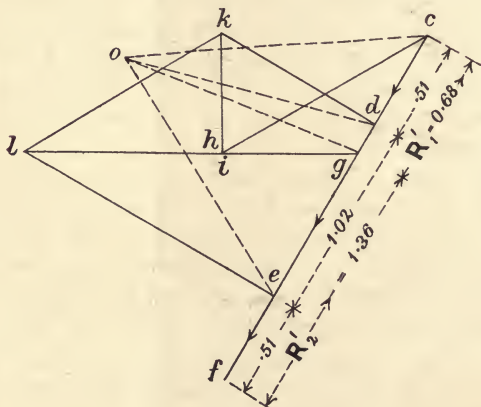


Fig. 20.

The part of the reaction  $R_2'$  which is due to  $ef$  is  $fe$  equal and opposite to  $ef$ . To find  $R_1'$ , and the remainder of  $R_2'$ , take a pole  $o$  and construct the force polygon by drawing  $oc$ ,  $od$ ,  $oe$ , and the corresponding funicular  $a\beta\gamma\delta$  (Fig. 18). Then in the force diagram (Fig. 20) draw  $og$  parallel to the closing line  $ad$ . The reactions are

$$\text{at joint 1, } gc = 0.68 \text{ ton} = R_1',$$

$$\text{,, ,, 2, } fg = 1.36 \text{ tons} = R_2'.$$

The *stress diagram* is begun by considering joint 5. Draw  $el$  and  $gl$  parallel to the bars  $EL$  and  $GL$ , then  $glefg$  is the reciprocal of the joint 5, and  $gl$ ,  $le$  represent the stresses due to wind pressure in bars  $GL$  and  $LE$ . Following the sense of direction indicated by  $R_2'$ , it is seen that the stress  $gl$  acts *from* the joint 5, and is therefore *tensile*, while  $le$  acts *towards* the joint and is *compressive*. Proceeding similarly for all the other joints, the diagram finally closes without a parallel to the bar  $HI$ . This means that for a wind blowing from the right bar  $HI$  is superfluous. If the wind blew from the left  $HI$  would be the active bar and  $KL$  would be superfluous.

The stresses due to the wind blowing from the right are found to be

In bars  $BH$  and  $CI$ ,  $c$ ,  $\bar{hi} = 1.18$  tons compressive

Bar $EL$ ,	$el = 1.48$	„	„
„ $KD$ ,	$kd = 0.90$	„	„
„ $HI$ ,	0		
„ $KL$ ,	$kl = 1.28$	„	„
„ $IK$ ,	$ik = 0.60$	„	tensile
„ $HG$ ,	$hg = 0.68$	„	„
„ $LG$ ,	$lg = 1.75$	„	„

By adding these with their proper sign to the stresses due to the vertical loads we get the resultant stresses.

If the wind blows from the left, the corresponding stress diagram will be similar to that in Fig. 20, but the stresses in those bars which are counterparts in the two halves of the frame will be interchanged. The two halves of the truss are usually symmetrical, the members being designed to resist the *maximum stresses* to which they can be subjected.

EXAMPLE II. ROOF TRUSS OF WHICH THE FRAME DIAGRAM IS SHOWN IN FIG. 21, FIXED AT BOTH ENDS.

<i>Data.</i>	Span between centres of bearings ...	40 feet.
	Distance between principals ...	12 feet.
	Inclination of rafters with horizontal ...	30°
	Height of horizontal tie above supports =	2 feet.
<i>Loads.</i>	Dead load per square foot of horizontal area	22 lbs.
	Pressure of horizontal wind per square foot of vertical surface ...	30 lbs.

In the last example *separate* stress diagrams were drawn for the fixed vertical loads and the wind loads. This was done to illustrate the general principles, but the most usual and quickest method, as adopted in this example, is to draw a single stress diagram to represent the combined effects of the dead and live loads.

*Dead load.* Total vertical load on one bay of roof

$$40 \times 12 \times 22 \text{ lbs.} = 10560 \text{ lbs.} = 4.72 \text{ tons.}$$

$$\text{Load at joints 1 and 5} = \frac{1}{8} 4.72 = 0.59 \text{ ton.}$$

$$\text{Load at joints 2, 3 and 4} = \frac{1}{4} 4.72 = 1.18 \text{ tons.}$$

*Reactions.* The reaction and load at *each support* have an *effective* reaction equal to their difference. This effective reaction is that due to the other loads on the truss, consequently the loads at the supports may be entirely omitted from consideration. Thus the reactions at 1 and 5 =  $\frac{1}{2} 3.54 = 1.77$  tons.

*Live load.* The intensity of normal wind pressure (see table Art. 10) is

$$\frac{30}{40} \times 26.4 = 19.8 \text{ lbs. per square foot.}$$

Total wind pressure acting on one principal is

$$12 \times 20 \sec 30^\circ \times 19.8 \text{ lbs.} = 5488 \text{ lbs.} = 2.44 \text{ tons.}$$

Assume the wind blows from the left, then

$$\text{Load at joints 1 and 3} = 0.61 \text{ ton.}$$

$$\text{,, ,, 2} = 1.22 \text{ tons.}$$

*Reactions.* The truss being fixed at the ends, the reactions are parallel to the normal wind pressure.

Their values are found from a funicular polygon drawn exactly as in last example (see Figs. 22 and 21).

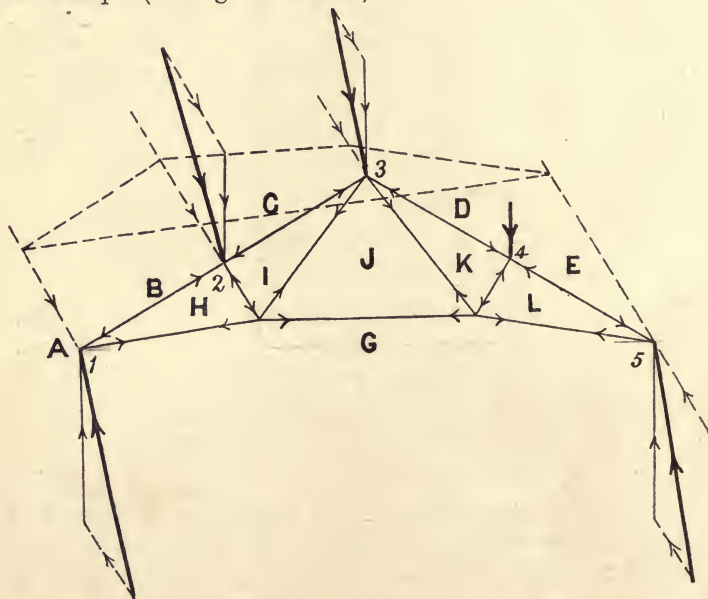


Fig. 21.

The values of the effective reactions are

at 1, 1.01 tons : at 5, 0.82 ton.

Now find the *resultant loads* acting at the joints 2 and 3 (Fig. 21) by compounding the vertical and wind loads. Similarly at joints 1 and 5 find the resultant upward reactions.

*Stress diagram* (Fig. 23). First construct the polygon of loading and supporting forces  $bcdgcb$ ; this must close.

Consider joint 1. Draw  $bh$  parallel to  $BH$  and  $gh$  parallel to  $GH$ .

Next for joint 2; draw  $ci$  and  $hi$  parallel respectively to  $CI$  and  $HI$ ; then  $bcihb$  is the reciprocal for joint 2.

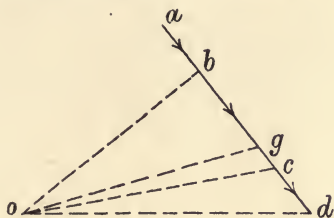
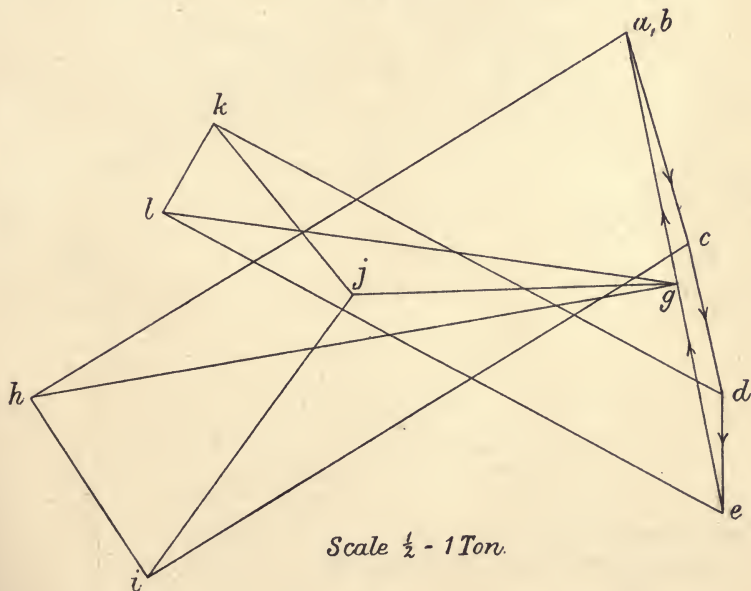


Fig. 22.



Scale  $\frac{1}{2}$  - 1 Ton.

Fig. 23.

This process may be continued by passing to the remaining joints in succession, in such order that at each joint only *two forces remain to be determined*. The complete stress diagram is shown in Fig. 23.

Table of stresses. Wind from the left.

Member	Stress compressive	Member	Stress tensile
BH	7.40	GH	6.90
CI	6.70	GJ	3.40
DK	6.10	GL	5.42
EL	6.65	IJ	3.71
HI	2.21	JK	2.28
KL	1.10		

As the wind may blow from the right, corresponding members must be designed to resist the *maximum stress* to which they can be subjected.

EXAMPLE III. ROOF TRUSS FIXED AT ONE END, AND FREE AT THE OTHER, STRESSES DUE TO WIND PRESSURE.

Let the roof truss (Fig. 24) be fixed at one end, and freely supported on rollers at the other end.

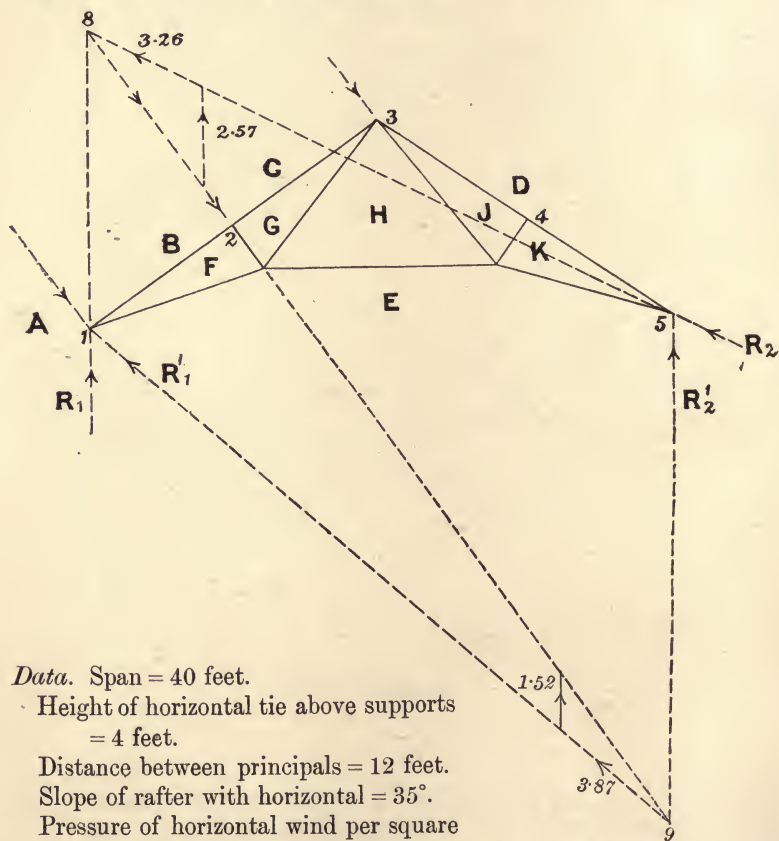


Fig. 24.

Data. Span = 40 feet.

Height of horizontal tie above supports  
= 4 feet.

Distance between principals = 12 feet.

Slope of rafter with horizontal =  $35^\circ$ .

Pressure of horizontal wind per square  
foot of vertical surface = 50 lbs.



Set off  $ab = 1\frac{1}{4}$  tons,  $bc = 2\frac{1}{2}$  tons,  $cd = 1\frac{1}{4}$  tons; draw  $de$  parallel and equal to  $R_2$ , i.e. 3.26 tons, then  $ea$  should be vertical and equal to  $R_1$ , i.e. 2.57 tons. The polygon  $abcdea$  is the polygon of loading and supporting forces and must close. The stress diagram is completed as in former examples. The points  $j, k$ , of stress diagram coincide as there is no stress in bar  $JK$ ; which is evident if we consider the equilibrium of joint 4, as there is no external load at this joint, and the stresses in  $DJ$  and  $DK$  are equal and opposite and in the same straight line.

*Case II.* Wind on the *fixed* side.

Keeping the wind on the same rafter, the end 1 of roof truss is now *fixed*, and the end 5 *free*.

*Reactions.* (Fig. 24.) The reaction at 5 is vertical. Produce the resultant normal wind pressure acting at centre of rafter 13 to meet the vertical through 5 at 9. Then 91 is the direction of the other reaction. The values of the reactions are found as in Case I. to be

$$R_1' = \text{reaction at 1} = 3.87 \text{ tons.}$$

$$R_2' = \text{reaction at 5} = 1.52 \text{ tons.}$$

*Stress diagram. Case II.* Fig. 26.

First draw the polygon of loading and supporting forces  $abcdea$ , in which  $ab, bc, cd$  are the wind loads as before;  $de$  and  $ea$  the reactions in direction and magnitude. The stress diagram can now be completed as explained before. That the points  $j$  and  $k$  coincide is a check on the accuracy of the work.

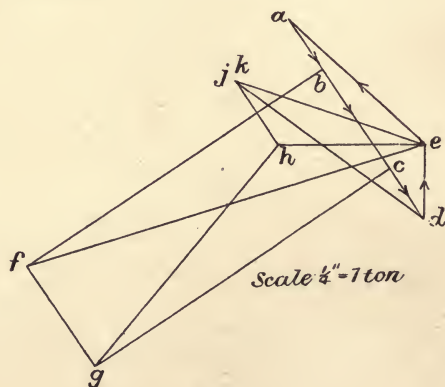


Fig. 26.

The following table gives the stresses in the members (by scale measurement) for the two cases :

Member	Case I.		Case II.	
	Stress in tons	Compression or tension	Stress in tons	Compression or tension
<i>BF</i>	4.32	<i>C</i>	7.30	<i>C</i>
<i>CG</i>	4.32	<i>C</i>	7.30	<i>C</i>
<i>FE</i>	2.96	<i>T</i>	8.55	<i>T</i>
<i>GH</i>	3.78	<i>T</i>	6.00	<i>T</i>
<i>HE</i>	0.97	<i>C</i>	3.10	<i>T</i> ✓
<i>FG</i>	2.50	<i>C</i>	2.50	<i>C</i>
<i>HJ</i>	0.55	<i>C</i>	1.73	<i>T</i> ✓
<i>JD</i>	1.92	<i>C</i>	5.80	<i>C</i>
<i>DK</i>	1.92	<i>C</i>	5.80	<i>C</i>
<i>KE</i>	1.36	<i>C</i> ✓	4.32	<i>T</i> ✓
<i>JK</i>	0		0	

## EXAMPLE IV.

The roof truss in Fig. 27 presents a difficulty which it is well to point out.

*Data.* Span between bearings = 60 feet.

Rise of horizontal tie above line of supports = 4 feet.

Apex of roof above line of supports = 20 feet.

Dead load at joints 2, 3, 4, 5, ... = 3 tons.

Vertical reaction at each abutment =  $10\frac{1}{2}$  tons.

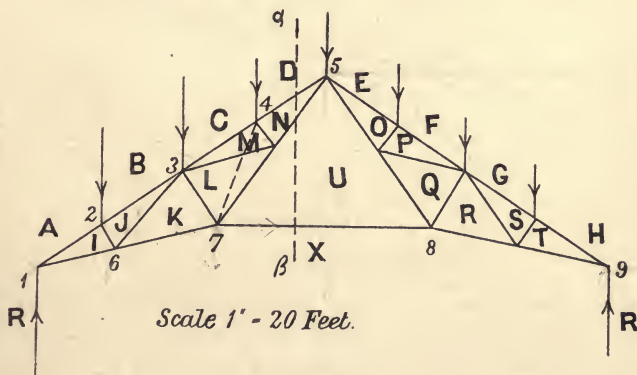


Fig. 27.

*Stress diagram.* (Fig. 28.) Commencing as usual at the left support, the force polygons for joints 1, 2, and 6 are easily constructed, and the stresses *ai*, *bj*, *xi*, *xk*, *kj*, *ij* obtained. Now at each of the joints 3 and 7 there are three stresses unknown, and as

the resultant of the known forces cannot be resolved in more than two given directions, the problem would appear indeterminate. The difficulty may be overcome by finding independently the stress in  $XU$ . This can be got very simply by the *method of sections*. Take a section  $\alpha\beta$  cutting the bars  $DN$ ,  $NU$ , and  $UX$ . The portion of the truss to the left of this section is acted upon by the three downward loads; the upward reaction; and the stresses in the three bars cut by the section. The sum of the moments of all these forces about any axis in their plane must be zero. Take this axis at the intersection of  $DN$  and  $NU$ , so that the moments of the forces acting in these lines must be zero. Then calling  $ux$  the stress in  $UX$ , we have

$$\begin{aligned}\overline{ux} \times 16 &= 10\frac{1}{2} \times 30 - 3(7\frac{1}{2} + 15 + 22\frac{1}{2}) \\ &= 315 - 135 = 180.\end{aligned}$$

Therefore  $\overline{ux} = 11.25$  tons.

Having found the stress in  $UX$ , that in  $KL$  can be found from the equilibrium of joint 7, and the stress diagram completed as in Fig. 28.

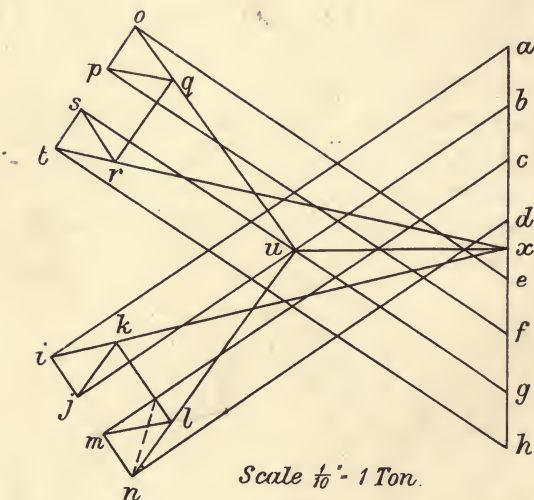


Fig. 28.

Another method is to remove the members  $LM$  and  $MN$  temporarily, and substitute for them the dotted diagonal shown in Fig. 27; then draw the stress diagram as usual till the stress in  $UX$  is found. The original bracing is now restored, and the stress diagram completed.

## 11. Method of Sections.

Suppose a frame to be divided into two segments by a plane section. Then, considering the separate equilibrium of each segment,

we see that *the stresses in the bars (or members) cut by the section must be in equilibrium with the external forces acting on the segment on either side of the section.*

As the forces are supposed to act in one plane, the conditions of equilibrium may be expressed analytically thus :

$$\Sigma (X) = 0, \Sigma (Y) = 0, \text{ and } \Sigma (M) = 0$$

where

$\Sigma (X)$  is the algebraic sum of the horizontal components;

$\Sigma (Y)$  is the algebraic sum of the vertical components;

$\Sigma (M)$  is the algebraic sum of the moments of the forces with respect to any axis.

The stresses can be determined by the solution of these three equations, if *not more than three bars* are cut by the section.

EXAMPLES OF THE APPLICATION OF METHOD OF SECTIONS TO A ROOF TRUSS.

Roof 30 ft. span (Fig. 29) carried on king trusses 10 feet apart, the rafters and struts of which are inclined at  $33^\circ$  with the horizontal. The vertical load is taken at 30 lbs. per square foot of horizontal surface covered.

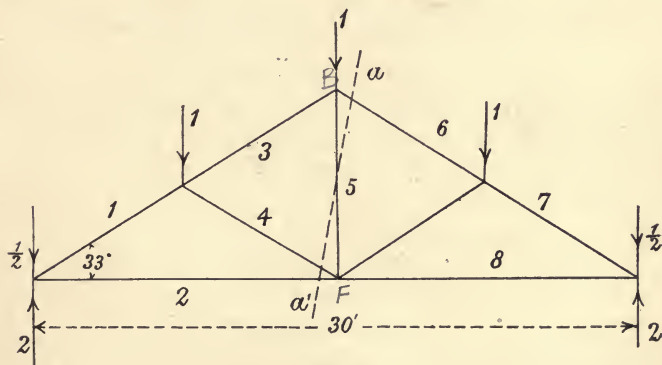


Fig. 29.

Determine the stresses in each bar of the truss.

The total vertical load

$$= 30 \times 10 \times 30 = 9000 \text{ lbs.}$$

$$= 4 \text{ tons nearly}$$

of which  $\frac{1}{2}$  ton is carried at each of the points A and C, and 1 ton at each of the points D, B, E.

Omitting the loads carried by the supports, the effective reactions are each equal to 1.5 tons.

Let  $S_1, S_2, S_3, \dots$  be the stresses in bars 1, 2, 3, ... of the frame.

To find  $S_1$  and  $S_2$ . At a vertical section cutting bars 1 and 2, we have, resolving vertically and horizontally,

$$1.50 = S_1 \sin 33^\circ,$$

$$S_2 = S_1 \cos 33^\circ.$$

Therefore 
$$S_1 = \frac{1.50}{0.545} = 2.75 \text{ tons, compression,}$$

$$S_2 = 2.75 \times 0.838 = 2.3 \text{ tons, tension.}$$

To find  $S_3$  and  $S_4$ . Take a section cutting bars 3, 4, 2; then the stresses  $S_3$ ,  $S_4$ ,  $S_2$  in the members 3, 4, 2 must balance the external forces of  $1\frac{1}{2}$  tons and 1 ton in order that equilibrium may be preserved. Take moments about  $F$  where the bars 2 and 4 meet.

$$S_3 \times 15 \sin 33^\circ - S_3 \times BF \cos 33^\circ = 1.5 \times 15 - 1 \times 7.5 = 15 \text{ ft. tons}$$

but 
$$BF = 15 \times \tan 33^\circ = 15 \times .65 = 9.75 \text{ feet.}$$

Therefore

$$S_3 = \frac{15}{9.75 \times 0.838} = 1.8 \text{ tons compression.}$$

Resolving horizontally,

$$S_3 \cos 33^\circ + S_4 \cos 33^\circ - S_2 = 0.$$

Therefore

$$S_4 = \frac{S_2 - S_3 \cos 33^\circ}{\cos 33^\circ} = \frac{2.3 - 1.8 \times 0.838}{0.838} \\ = 0.95 \text{ tons compression.}$$

To find  $S_5$ . Take an approximately vertical section  $aa$ , passing to the right of  $B$ , and to the left of  $F$ .

$S_6 = S_3$  from symmetry of roof and loading.

Resolving vertically,

$$S_3 \sin 33^\circ + S_4 \sin 33^\circ - S_5 - 1.5 - 1 - 1 = 0,$$

$$S_5 = (S_3 + S_4) \sin 33^\circ - 0.5$$

$$= 2.75 \times 0.545 - 0.5$$

$$= 1.0 \text{ ton tension.}$$

The truss being loaded symmetrically, the stress in corresponding bars of the two halves is the same.

## 12. Girders.

A girder consists of

(a) An upper member arranged in a straight or polygonal line, and called the "*top boom*" or *top chord*.

(b) A lower member similarly formed, and called the "*bottom boom*" or *bottom chord*.

(c) A series of members, either all inclined, or some vertical and others inclined, connecting the two chords, and forming with them a series of triangles. These are called the "*web*."

A girder of uniform depth has both chords straight and parallel. A girder of variable depth has usually the top chord curved or polygonal, and the bottom chord straight; the greatest depth being at the centre.

Girders of uniform depth belong to two principal classes :

(a) That in which the web-bars make alternately equal and opposite angles with the vertical, forming with the chords a system of *isosceles triangles* (Fig. 30). This is called a *Warren girder*.



Fig. 30.

(b) That in which the web-bars are alternately vertical and inclined, forming with the chords a system of *right-angled triangles* (Fig. 31).



Fig. 31.

The portion of either chord comprised between two adjacent joints is called a "*bay*" or "*panel length*"; each corresponding division of the girder is called a "*panel*."

*Loads on girders.* (a) *The dead load* due to the weight of the flooring, trusses, and lateral bracing; it is usually taken as uniformly distributed.

(b) *The live load* which travels over the bridge, such as trains.

The live load is sometimes taken as uniformly distributed over the span, but is now generally taken as consisting of two locomotives with their tenders (the weight of these being concentrated upon the wheels), followed by a uniform train load.

The maximum *chord stresses* due to a live load occur when every panel point is loaded. The maximum stress in a web member is produced when the live load covers the longer segment of the span, and the minimum stress when the smaller segment is loaded. The maximum and minimum stresses in the web members due to both dead and live loads are obtained by adding, with their proper signs, the dead load stresses to each of the corresponding live load stresses.

### 13. Bridge Truss with Horizontal Chords. Dead load stresses.

A Pratt truss of eight panels (Fig. 32). Span 168 feet. Length of bay 21 feet. Depth 24 feet. Dead load at each panel point of the lower chord 8 tons.

On the vertical line of loads (Fig. 33) set off  $bj$  by scale equal to  $8 \times 7 = 56$  tons, and divide into seven equal parts  $bc, cd, \dots hj$ . Bisect  $bj$  in  $a$ , then  $ja$  and  $ab$  are the effective reactions.

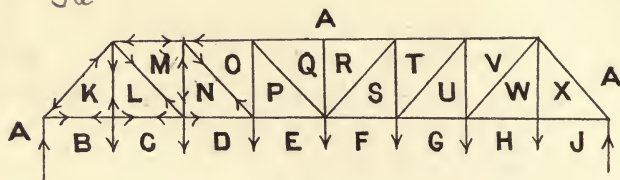


Fig. 32.

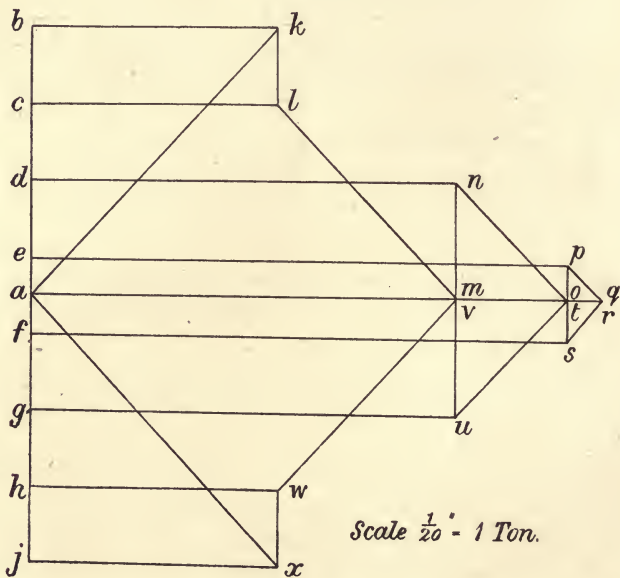


Fig. 33.

First, consider the equilibrium of joint  $ABK$ . The triangle of forces for this joint is  $abk$ , and as  $ab$  is upwards, the other forces must act in the same direction round the triangle; therefore the stress in  $BK$  acts from the joint and is tension; the stress in  $AK$  acts towards the joint and is compression. Similarly for the upper joint  $AMLK$ , we get  $ak$  compression,  $kl$  tension,  $lm$  tension, and  $ma$  compression. The stress diagram is completed by considering the joints alternately on the bottom and top chords. The upper and lower halves of the stress diagram will be symmetrical about  $aq$ . This diagram (Fig. 33) shows, that the tensile stress in the lower chord and the compressive stress in upper chord increase from the end toward the centre of the truss; while the stresses in the web members increase from the middle

towards the ends of the truss. The diagonals are all in tension except  $AK$  and  $AX$ . The verticals are all in compression except  $KL$  and  $WX$ , which merely transmit the loads  $BC$  and  $HJ$  to the top chord. The stress in vertical  $QR$  is zero.

As the polygon  $adnm$  is a rectangle, the tension ( $dn$ ) in  $DN$  is equal to the compression ( $am$ ) in  $AM$ . Also since  $ao$  is equal to  $am$  plus  $mo$ , the stress in  $AO$  = stress  $AM$  plus the horizontal component of the stress in  $NO$ .

The stresses in the different members of the truss can be got from Fig. 33 by scale. The stresses due to live (moving) loads will be considered later.

### EXERCISES.

1. The derrick crane in Fig. 34 carries a load of 5 tons suspended from  $A$ , at a distance of 60 feet from the axis of the post  $BC$ . Find the supporting forces at  $C$  and  $D$ , and the stress in each member.

Ans. Reaction at  $C$  = + 16.81 tons

„ „  $D$  = - 11.81 „

Stress in  $AB$  = 13.66 tons tension

„  $AC$  = 16.73 „ compression

„  $BC$  = 4.98 „ compression

„  $BD$  = 16.7 „ tension

$CD$  = 11.81 „ compression.

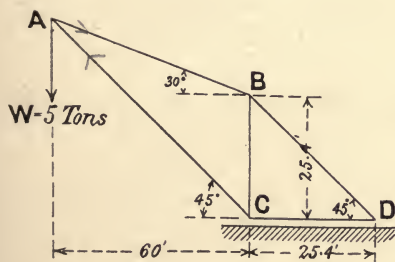


Fig. 34.

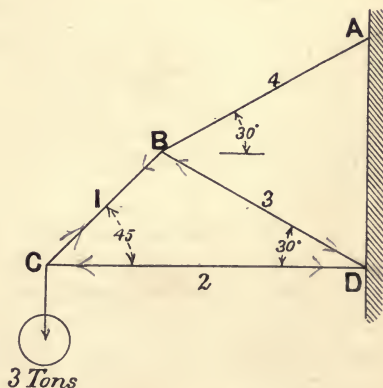


Fig. 35.

2. The framed cantilever in Fig. 35 is fixed at  $A$  and  $D$ , and carries at  $C$  a vertical load of 3 tons. Find the stress in each member of the truss; and determine the vertical and horizontal components of the reactions at  $A$  and  $D$ .

Ans. Stress in  $BC = 4.24$  tons tension  
 „  $CD = 3$  „ compression  
 „  $BD = 1.27$  „ compression  
 „  $AB = 4.7$  „ tension.  
 Vertical component at  $A = 2.36$  tons  
 „ „ „  $D = 0.64$  „  
 Horizontal „ „ „  $A = 4.1$  „  
 „ „ „  $B = 4.1$  „

3. The roof truss in Fig. 36 carries vertical loads as follows :

At  $A$  and  $C$ ,  $0.25$  ton. At  $D$  and  $E$ ,  $0.84$  ton. At  $B$ ,  $0.50$  ton.

Find the angle  $FAJ$ , and the stress in each bar of the truss by the method of sections.

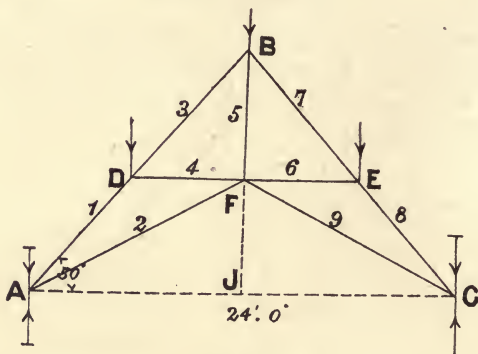


Fig. 36.

NOTE.  $D$  and  $E$  are at the middle points of  $AB$  and  $BC$  respectively and  $DFE$  is horizontal.

Ans. Angle  $FAJ = 30^\circ 48'$ .

Stresses are : Compressive, in 1 and 8,  $2.84$  tons ; in 3 and 7,  $1.74$  tons ; in 4 and 6,  $0.71$  ton. Tensile, in 2 and 9,  $2.12$  tons ; in 5,  $2.15$  tons.

4. Fig. 37 represents a portion of a bridge truss cut by a plane  $aa_1$ . The span of truss is  $235$  feet, divided into  $10$  bays, each  $22\frac{1}{2}$  ft. long. Dead load carried at each panel point of lower chord is  $18$  tons.

$BG = 19$  ft. ;  $CH = 22.1$  ft. ;

$DI = 24.2$  ft.

Find by method of sections the stress in each of the three members cut by the section  $aa_1$ .

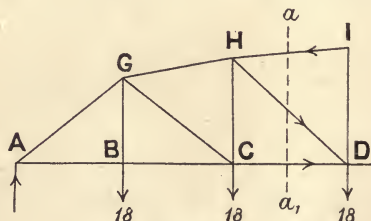


Fig. 37.

Ans.  $HI = 180$  tons C. ;  $HD = 46.3$  tons T. ;  $CD = 146.6$  tons T.

5. A roof truss 30 ft. span in Fig. 38 carries a vertical load of 4 tons, distributed as follows : at joints 1 and 2, 0.27 ton ; at joint 3, 0.53 ton ; at each of the joints 4, 5, 6, 7, 0.73 ton. (a) Find the

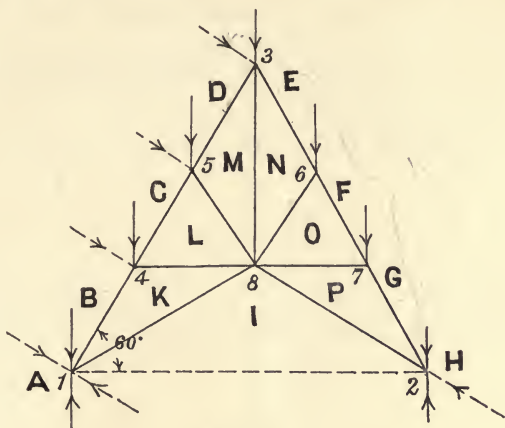


Fig. 38.

stress in each member due to these vertical loads. The truss also supports a normal wind load of 4 tons, distributed at joints 1 and 3, 0.54 ton ; at joints 4 and 5, 1.46 ton. (b) Find the stress in each member due to the wind.

*Ans.* NOTE. Compressive stress marked +  
Tensile stress marked -

Vertical load		Wind load	
Member	Stress in tons	Member	Stress in tons
<i>BK, PG</i>	+3.0	Reaction at	2=4
<i>CL, FO</i>	+2.16	Reaction at	1=0
<i>DM, EN</i>	+1.74	<i>BK</i>	-0.95
<i>KL, OP</i>	+0.42	<i>CL</i>	-1.79
<i>LM, ON</i>	+0.43	<i>DM</i>	-0.95
<i>KI, IP</i>	-1.73	<i>MN</i>	+1.1
<i>MN</i>	-2.47	<i>ML</i>	+1.68
		<i>LK</i>	+1.68
		<i>KI</i>	+1.1

Combine the stresses so as to find the maximum stress in each member due to both loads.

6. An overhanging roof for a railway station platform, 50 ft. span (Fig. 39), is supported on a wall at one end, and on an intermediate column 30 ft. from the wall. It carries fixed vertical loads distributed as follows :

At  $A$ , 0.16 ton;  $B$ , 0.44 ton;  $C$ , 0.44 ton;  $D$ , 0.32 ton;  $E$ , 0.46 ton;  $F$ , 0.37 ton;  $H$ , 0.46 ton;  $L$ , 0.16 ton.

Find the reactions, and the stress in each member of truss.

*Ans.* Reaction at wall = 0.4 ton.

Reaction at column = 2.41 tons.

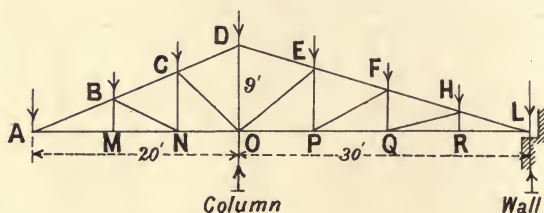


Fig. 39.

Verticals  $BM$  and  $HR$  are required only to prevent sagging of the members  $AN$  and  $QL$ .

$AB = -0.41$ ;  $BC = -0.94$ ;  $CD = -1.48$ ;  $AN = +0.37$ ;  $NO = +0.86$ ;  $BN = +0.54$ ;  $CN = -0.22$ ;  $CO = +0.68$ ;  $DO = +1.33$ ;  $DE = -1.40$ ;  $EF = -0.65$ ;  $FH = +0.05$ ;  $HF = +0.81$ ;  $OP = +0.63$ ;  $PQ = -0.04$ ;  $QL = 0.78$ ;  $EO = +0.95$ ;  $FP = +0.78$ ;  $HQ = +0.78$ .

7. In the last example, if the end  $L$  is firmly fixed to the wall, and the column is fixed at the foot, find the stress in each member due to normal wind loads at the joints as follows: at  $A$  and  $D$ , 0.21 ton; at  $B$  and  $C$ , 0.58 ton.

*Ans.* Reaction at wall = 0.43 ton.

Reaction at column = 2.01 tons.

$AB = -0.47$ ;  $BC = -0.98$ ;  $CD = -1.5$ ;  $AN = +0.52$ ;  $NO = +1.22$ ;  $BN = +0.78$ ;  $CN = -0.32$ ;  $CO = +0.95$ ;  $DO = +1.19$ ;  $OL = +1.11$ ;  $DL = -1.35$ .

8. If, in the last example, the column is hinged at the foot so that the reaction there is vertical, find the stress in each member due to the wind loads.

*Ans.* Reaction at wall = 0.75 ton.

Vertical reaction at column = 1.83 tons.

$AB = -0.47$ ;  $BC = 0.98$ ;  $CD = 1.51$ ;  $AN = +0.52$ ;  $NO = +1.22$ ;  $BN = +0.78$ ;  $CO = +0.95$ ;  $CN = -0.32$ ;  $DO = +1.19$ ;  $OL = +1.95$ ;  $DL = -1.35$ .

9. A roof truss fixed at the ends 50 ft. span as in Fig. 40, the rafters of which make an angle of  $30^\circ$  with the horizontal, carries vertical loads of  $AB$ ,  $BC$ ,  $CD$ ,  $DE$ ,  $EF$  each equal to  $1\frac{1}{2}$  tons, also wind loads

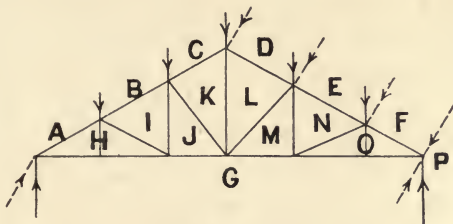


Fig. 40.

on the right, viz.  $CD = 0.56$  ton,  $DE = 1.12$  tons,  $EF = 1.12$  tons,  $FP = 0.56$  ton. Determine the stress in each member of the truss due to both loads, having first found the reactions due to the wind graphically.

*Ans.*  $AH = +9.4$ ;  $BI = +8.0$ ;  $CK = +6.2$ ;  $DL = +6.1$ ;  $EN = +8.2$ ;  $FO = +10.20$ ;  $GH = -7.4$ ;  $GJ = -6.0$ ;  $GM = -7.3$ ;  $GO = -9.5$ ;  $HI = +1.5$ ;  $IJ = 0.7$ ;  $KJ = +2$ ;  $KL = -4.2$ ;  $LM = +3.5$ ;  $MN = -1.25$ ;  $NO = +2.5$ .

10. Roof truss as in Fig. 41; span 77 ft.; slope of rafter  $30^\circ$ ; rise of roof 22.23 ft. Each rafter is divided into three equal parts, and

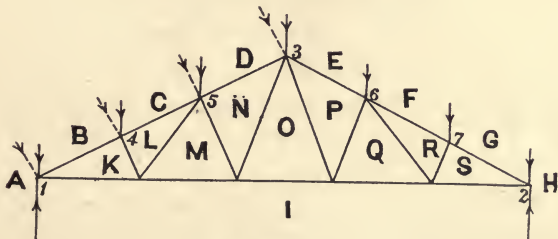


Fig. 41.

the main tie into five equal bays. The total *vertical* load on truss is 6.2 tons distributed at the joints as under:

At 1 and 2, 0.41 ton; at 3, 0.83 ton; at 4, 5, 6, and 7, 1.14 tons.

The resultant wind pressure is 7.15 tons assumed acting on the left, divided between the joints thus:

At 1 and 3, 0.95 ton; at 4 and 5, 2.62 tons.

*The truss is fixed at 1, and free at 2.*

Determine the maximum stress in each member due to both loads,

remembering that counterparts in opposite halves of the truss must each have the same strength, as the wind may blow from either side.

(Compressive stress + ; Tensile -.)

	Tons		Tons
<i>Ans.</i> <i>BK</i> and <i>GS</i>	+ 12·01	<i>CL</i> and <i>FR</i>	+ 12·2
<i>DN</i> and <i>EP</i>	+ 8·8	<i>KI</i> and <i>SI</i>	- 13·6
<i>MI</i> and <i>QI</i>	- 9·95	<i>OI</i>	- 6·3
<i>KL</i> and <i>RS</i>	+ 3·7	<i>LM</i> and <i>QR</i>	- 4·3
<i>MN</i> and <i>PQ</i>	+ 5·6	<i>NO</i> and <i>OP</i>	- 5·6

11. A Warren girder as in Fig. 42, of span 60 ft., carries a load of 20 tons at each joint of the bottom chord. Find the stress in each member of the girder.

The bracing is inclined at  $60^\circ$  to the horizontal. The lower chord is divided into six equal bays of 10 ft. in length.

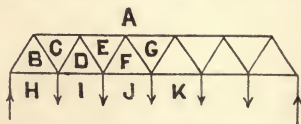


Fig. 42.

*Ans.*  $AB = + 57·8$  ;  $BC = - 57·8$  ;  $CD = + 34·5$  ;  $DE = - 34·5$  ;  
 $EF = + 11·5$  ;  $FG = - 11·5$  ;  $AC = + 57·8$  ;  $AE = + 92·4$  ;  $AG = + 103·8$  ;  
 $HB = - 28·9$  ;  $IC = - 75·0$  ;  $JF = - 98·0$ .

12. A Fink truss as in Fig. 43, span 20 ft., divided into four equal bays of 5 ft., depth 3 ft., carries loads  $AB = EF = 1·97$  tons;

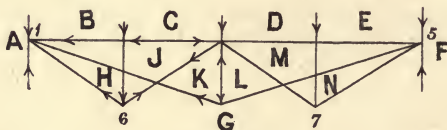


Fig. 43.

$BC = DE = 5·7$  tons ;  $CD = 4·6$  tons. Find the stress in each member graphically.

NOTE. Member  $\overline{16}$  is  $HK$ , and  $\overline{57}$  is  $NL$  ; as the truss consists of a primary truss and two secondary trusses. Commence the stress diagram at joint 6 where there are only two unknown stresses, as the stress in  $HJ$  is equal to the vertical load  $BC$ .

*Ans.*

$BH = CJ = DM = EN = + 21·9$  tons     $KG = LG = - 17·9$  tons  
 $HK = JK = ML = NL = - 5·5$  tons     $HJ = MN = + 5·7$  tons  
 $KL = + 10·3$  tons.

13. A girder 60 ft. span carries loads of 15 tons, 14 tons, 20 tons, 24 tons and 16 tons at points 17 ft., 25 ft., 32 ft., 47 ft., 56 ft. respectively from the left support. Draw by means of the vector and link polygons the bending moment diagram. Find the scale of this diagram, and give the value of the maximum bending moment.



## CHAPTER II.

### STRESS AND STRAIN.

14. IN designing a structure we must first determine the forces which tend to produce deformation. These are called the *external loads*, or loads simply. They are measured in force units; usually lbs. or tons. Knowing the loads, it is necessary to calculate the stresses produced by them in the different parts of the structure. Further, we must have an accurate knowledge of the properties of the materials employed as to strength and elasticity.

STRESS AND STRAIN. A body which is acted on by external forces is said to be *strained* or *deformed*. The force exerted in the interior of the body, and which resists deformation, is called *stress*.

The state of strain is *simple* when the stress acts in *one* direction only; it is *compound* when *two or more* stresses are acting in different directions. There are five kinds of stresses, viz. :—

- (a) A longitudinal pull or tension.
- (b) A longitudinal push or compression.
- (c) Tangential or shearing stress, which tends to make the two surfaces of a section slide on one another.
- (d) Transverse stress.
- (e) Twisting or torsion.

Let  $AB$ , Fig. 44, be a bar circular in cross-section acted on by two equal and opposite axial forces  $F$ , both acting outwards, and constituting a state of tension. Imagine  $AB$  divided into two parts,  $C$  and  $D$ , by a plane at right angles to the axis. Then the force which

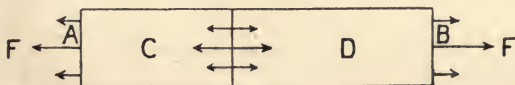


Fig. 44.

$C$  exerts on  $D$  must be in equilibrium with the force  $F$  on the end  $B$ . Similarly the force which  $D$  exerts on  $C$  must balance the force  $F$  on

the end *A*. Thus there exists at the section between *C* and *D* a *stress*, the total amount of which balances *F*. So *F* is the total stress on the section.

**INTENSITY OF STRESS.** Stress is usually measured by its *intensity* or stress per unit of area.

**TENSILE STRESS.** In Fig. 44 let *A* be the area of section of bar in square inches, then if the stress of *F* tons is uniformly distributed so that each square inch of area bears the same amount of stress, the *intensity of tensile stress* is

$$f = \frac{F}{A} \text{ tons per square inch ;}$$

or, *intensity of stress is the force per unit of area.*

**COMPRESSIVE STRESS.** If, as in Fig. 45, the forces *F* act inwards towards the bar they constitute *compression*, tending to cause failure by crushing, and either force is called a *compressive load*.

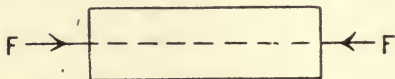


Fig. 45.

In this case, also, the compressive stress is

$$f = \frac{F}{A} \text{ tons per square inch.}$$

In both these cases of simple pull and push the following condition must be fulfilled in order to ensure uniformly-distributed stress, *i.e. the line of action of the resultant load must lie in the axis of the bar*, the axis being the line joining the centres of gravity of all the normal cross-sections.

**SHEARING STRESS.** If two equal and opposite forces *Q*, *Q* act parallel and tangential to a section *ab*, as in Fig. 46, the two portions of the bar separated by the section tend to slide past one another. The forces are called *shearing forces*, and produce a *shearing* or *tangential stress* on the section. If *A* is the area of the section the intensity of shearing stress is

$$q = \frac{Q}{A}.$$

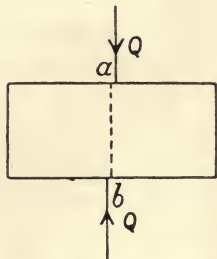


Fig. 46.

*Bending or transverse stress and torsion* will be considered later.

**STRAIN.** Strain is a change of shape produced by stress. If a bar is pulled by an axial load applied at the ends, the stress is tensile on cross-sections normal to the axis, and the strain consists of an elongation in length, accompanied by a diminution in section. If the

stress is compressive the strain consists of a shortening in length, accompanied by an increase in section. The strain in shear is simply a distortion by the sliding of one section on another in the direction of the shearing stresses.

### 15. Elasticity. Permanent set.—Elastic limit.

When a stress is applied to a body a deformation or strain is produced. The strain is said to be *elastic* if it disappears when the stress is removed. If, on the other hand, after the stress is removed the deformation still remains, the strain is *plastic*, and is called *permanent set*.

The limiting stress up to which the strain is elastic, and above which a permanent set is produced, is called the *elastic limit*. For a stress less than the elastic limit the strain is proportional to the stress; for a stress greater than the elastic limit the strain increases much more rapidly than the stress. Experiment shows that the limit of elasticity under repeated stresses of equal intensity, alternately tensile and compressive, tends to rise. The same result is also found when the stresses vary from a maximum stress to a minimum stress of the same kind.

### 16. Relation between Stress and Strain.

Let a prism of length  $L$  and cross-section  $A$  be stretched or compressed longitudinally by a force  $F$  applied at the ends, acting along the axis (Fig. 47).

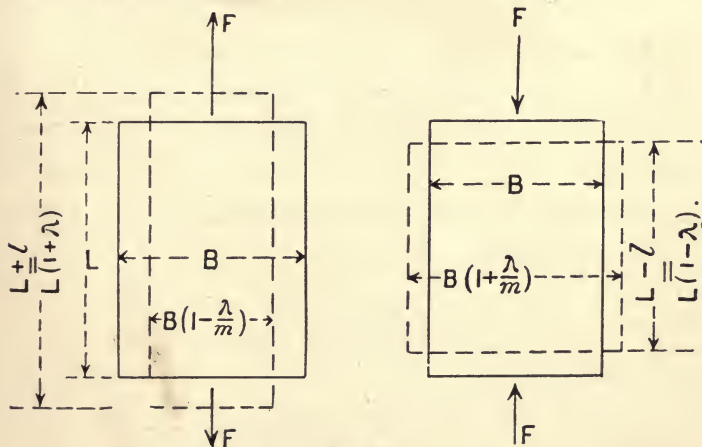


Fig. 47.

Let  $l$  be the extension or compression, *i.e.* the deformation.

The *intensity of stress* on cross-sections is

$$f = \frac{F}{A}.$$

The *strain* is measured by the ratio

$$\frac{l}{L},$$

that is the deformation per unit of original length.

Strain being a ratio of one length to another length is simply a number. It is not expressed in any unit.

The relation between stress and strain known as Hooke's law is, that within the elastic limits the *strain is proportional to the stress* producing it.

Thus 
$$\frac{l}{L} \propto f;$$

or 
$$\frac{l}{L} = \frac{f}{E}.$$

If the strain measured per unit of original length be denoted by  $\lambda$ , we have

$$f = E\lambda, \text{ or } \lambda = \frac{f}{E},$$

$E$  being a constant, which varies with different materials. It is called *Young's Modulus of Elasticity*. Within the elastic limits  $E$  has practically the same value for tension and compression. It is expressed in the same units as the intensity of stress.

For wrought iron and steel this modulus is about 13,000 tons per square inch; for cast iron about 6,000 tons per square inch.

### 17. Poisson's Ratio.

When a bar is extended by longitudinal pulling forces (Fig. 47) it contracts laterally. The *longitudinal strain* is  $+\lambda$  for extension, and the *lateral strain* is  $-\frac{\lambda}{m}$ . If the bar is compressed by pushing forces (Fig. 47) it expands laterally. The longitudinal strain is  $-\lambda$  for compression, and the lateral strain is  $+\frac{\lambda}{m}$ .

The ratio 
$$\frac{1}{m} = \frac{\text{lateral strain}}{\text{longitudinal strain}}$$

is called *Poisson's Ratio*.

For metals  $m$  has a value between 3 and 4.

### 18. Stress due to change of temperature.

If the temperature of a bar of length  $L$  is raised  $t^\circ$  its altered length becomes  $L(1 + \alpha t)$ ,  $\alpha$  being the coefficient of linear expansion. The elongation is  $Lat$ , and the strain is  $\alpha t$ . If a change of length is prevented the stress developed is  $E\alpha t$ .

### 19. Normal and tangential stress on an oblique plane.

Let  $BC$  (Fig. 48) be a bar acted on in the direction of its length by a force  $F$  uniformly distributed. Let  $A$  be the area of the normal cross-section.

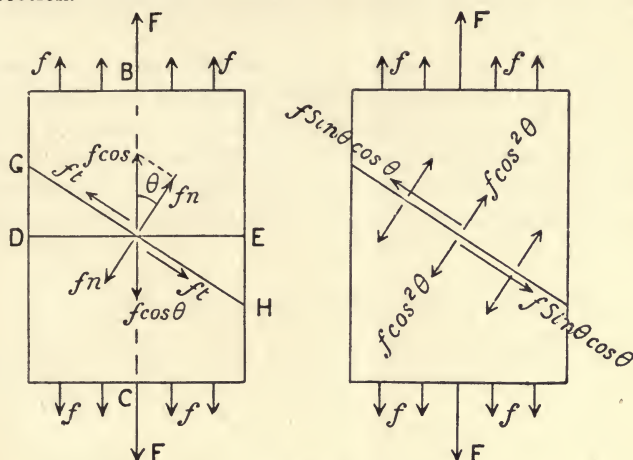


Fig. 48.

The intensity of stress on a section  $DE$  normal to the axis is

$$f = \frac{F}{A}.$$

Now take an oblique plane  $GH$ , whose normal is inclined at an angle  $\theta$  to the axis, and whose area  $= \frac{A}{\cos \theta}$ . The total stress on  $GH = F$ , and acts in the direction of the axis.

$$\begin{aligned} \text{The intensity of this stress on } GH &= \frac{F}{\text{area of } GH} = \frac{F}{A} \cos \theta; \\ &= f \cos \theta. \end{aligned}$$

The intensity of the normal component on  $GH = f \cos^2 \theta$ .

The intensity of the tangential component or shearing stress on  $GH$

$$= f \cos \theta \sin \theta = f \frac{\sin 2\theta}{2}.$$

The tangential or shearing stress is evidently a maximum when  $\sin 2\theta$  is a maximum, that is when  $2\theta = 90^\circ$ , or  $\theta = 45^\circ$ , and the intensity of maximum shearing stress will be

$$\text{Max. } q = \frac{f}{2}.$$

If we take a section at right angles to  $GH$  the intensity of the normal component on it

$$= f \sin^2 \theta.$$

The tangential component or shearing stress on it *is also*

$$= f \sin \theta \cos \theta ;$$

or, the shearing stresses on two planes at right angles are equal, and the planes of maximum shear are inclined at an angle of  $45^\circ$  to the direction of the stress.

In practical tests of tension and compression it is found that fracture does not take place on a surface inclined at an angle of  $45^\circ$  to the axis. This may be due to the normal component of pull in tension which diminishes the resistance to shearing, and the normal component of push which increases the resistance to shearing. Experiment shows that the inclination of the surfaces of shear to the axis of the piece is about  $35^\circ$  for tension, and  $55^\circ$  for compression.

## 20. Shearing Strain.—Modulus of Rigidity.

To consider the deformation due to shear, imagine a small cubical element of material (Fig. 49) to be fixed on the face  $AB$ , and acted on

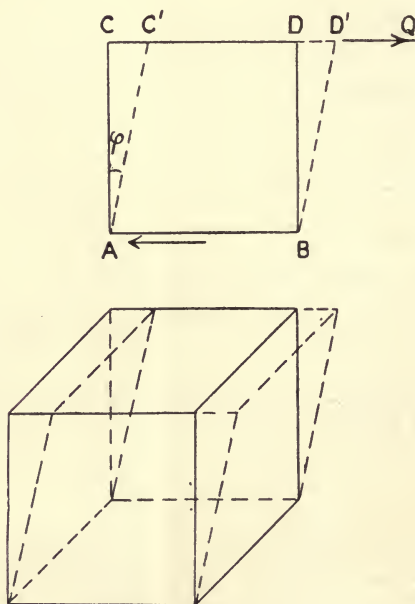


Fig. 49.

by a force  $Q$  along the top face  $CD$ . The cube will become distorted, being lengthened in one diagonal direction, and shortened equally in the other. The sides remain parallel.

The square face  $ABDC$  becomes distorted into a rhombus  $ABD'C'$ . The strain is measured by the change of angle, i.e.  $\phi$ , expressed in circular measure.

Within the limits of elasticity the deformation  $\phi$  is proportional to the intensity of shearing stress  $q$ , ( $\frac{q}{A}$ )

$$\text{i.e.} \quad q = C \cdot \phi;$$

$$\text{and} \quad C = \frac{q}{\phi} = \frac{\text{shearing stress}}{\text{shearing strain}},$$

where  $C$  is a constant called the *Modulus of Rigidity*. Its value for wrought iron is about 5,000 tons per square inch, and for steel about 5,200 tons per square inch. For cast iron it is about 2,000 tons per square inch.

## 21. Equality of Shearing Stress on planes at right angles.

Let  $ABCD$  (Fig. 50) be a small rectangular parallelopiped of unit thickness, the stresses on which are tangential stresses on two pairs of faces.

The total stress on the face  $AB = q_1 \cdot AB$ , which is equal in magnitude but opposite in direction to  $q_1 \cdot DC$ , the total stress on  $DC$ . The moment of this couple is

$$q_1 \cdot AB \cdot AD.$$

Similarly the total stresses on  $AD$  and  $BC$  form a couple of moment

$$q_2 \cdot AD \cdot AB.$$

For equilibrium these two couples must balance.

$$\text{Therefore} \quad q_1 AB \cdot AD = q_2 AD \cdot AB;$$

that is

$$q_1 = q_2.$$

Hence, at any point in a strained body, whatever the normal stresses may be, shearing stress in one direction is always accompanied by a shearing stress of equal intensity in a direction at right angles to the first. This proof is implied in Art. 19.

TWO EQUAL SHEARING STRESSES ON TWO PLANES AT RIGHT ANGLES TO ONE ANOTHER ARE EQUIVALENT TO A TENSILE STRESS, AND A COMPRESSIVE STRESS OF EQUAL INTENSITY TO THAT OF THE SHEARING STRESS, ON PLANES AT  $45^\circ$  TO THE DIRECTIONS OF THE SHEARING STRESS.

Imagine a small cubical element of side  $h$  of the material, subjected to shearing stress of intensity  $q$  on parallel faces  $AB$ ,  $DC$ , and  $AD$ ,  $BC$

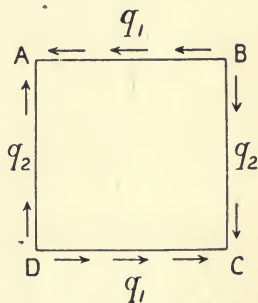


Fig. 50.

(Fig. 51). Consider the equilibrium of the small right prism whose base is  $ADB$ , cut off by the diagonal section  $DB$ .

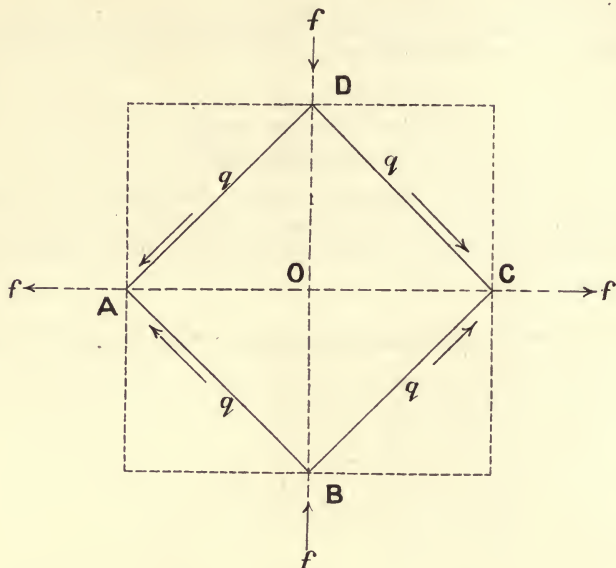


Fig. 51.

The forces acting on  $AB$  and  $AD$  are each  $= qh^2$ . Their resultant is  $qh^2\sqrt{2}$ , acting in the direction  $OA$ . To balance this there must be a *normal pull or tension* on the diagonal plane  $DB$ , acting in the direction  $OC$ . Call this force  $F$ , then

$$F = qh^2\sqrt{2};$$

or

$$\frac{F}{h^2\sqrt{2}} = q;$$

but  $h^2\sqrt{2}$  is the area of the plane  $DB$  on which  $F$  acts.

Therefore

$$f = q;$$

i.e. the intensity of the normal tensile stress on face  $DB$  is equal to the intensity of the shearing stress on each of the other two faces. Similarly, considering the equilibrium of the prism standing on the base  $ACB$ , cut off the other diagonal  $AC$ , the resultant of the forces on  $AB$  and  $BC$ , i.e.  $qh^2\sqrt{2}$ , acts in the direction  $BO$ ; consequently for equilibrium there must be a *normal pressure or compression* acting in the direction  $DO$ , and, as above, its intensity is

$$f = \frac{F}{h^2\sqrt{2}} = q.$$

The straining due to the shearing stresses is therefore the same as that produced by a pressure and tension of equal intensity upon planes at  $45^\circ$ .

## 22. Two normal Stresses at right angles to one another.

Consider a small cube of *unit side*, acted on by normal stresses of intensity  $f_1, f_2$  upon the faces  $AD, BC$ , and  $AB, DC$ .

Case I. Stresses of the same sign (Fig. 52).

The strains are

$$\text{Parallel to } AB, \quad \lambda_1 = \frac{f_1}{E} - \frac{f_2}{mE},$$

$$\text{Parallel to } AD, \quad \lambda_2 = \frac{f_2}{E} - \frac{f_1}{mE},$$

and the strain perpendicular to  $ABCD$ ,

$$\lambda_3 = -\frac{f_1}{mE} - \frac{f_2}{mE}.$$

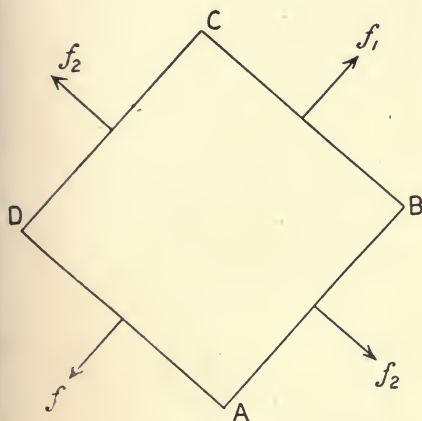


Fig. 52.

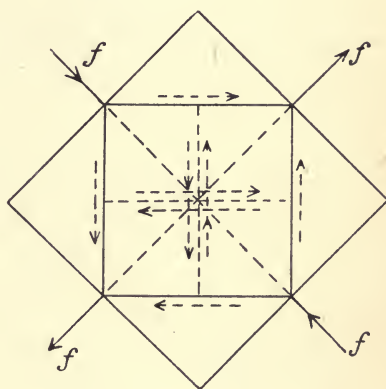


Fig. 53.

Case II. If the stresses are of *equal intensity*, but of *opposite sign* (Fig. 53), then

$$f_1 = -f_2 = f, \text{ say.}$$

This is a case of simple shearing stress on two planes inclined at  $45^\circ$  to the directions of the normal stresses, and *the strain*

$$\lambda_1 = -\lambda_2 = \frac{f}{E} \left( 1 + \frac{1}{m} \right),$$

$$\lambda_3 = 0.$$

The volume of the solid becomes  $(1 + \lambda_1)(1 - \lambda_1)(1)$ , which is equal to 1, if  $\lambda^2$  be neglected. So that the volume is inappreciably changed.

### 23. Relation between constants $E$ , $C$ , and $m$ .

An enclosed square (Fig. 54), drawn on the side of the original cube in Case II. of last article, will become distorted into a rhombus, one diagonal being lengthened, and the other shortened. Let  $X$  be the original length of the diagonal and  $x$  its extension; let  $Y$  be the length of the side of square and  $y$  the movement of the side.

From Fig. 54 the distortion can be most easily realised if we first imagine  $AB$  fixed, and  $CD$  moved to  $FG$  by the shear on  $CD$ ; then consider  $AF$  fixed and  $BG$  moved to  $EH$  by the shear on  $BG$ , which is the same as the shear on  $BD$ , as the angle  $DBG$  is very small.

Then from Fig. 54

$$\frac{x}{X} = \frac{y}{Y} = \frac{\phi}{2}.$$

$\frac{x}{X}$  is the tensile strain along one diagonal;

$\frac{y}{Y}$  is the compressive strain along the other diagonal

(the total shortening is  $y\sqrt{2}$ , the original length =  $Y\sqrt{2}$ );  
 $\phi$ , the change of angle, is the shearing strain.

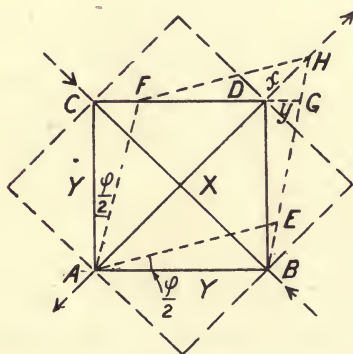


Fig. 54.

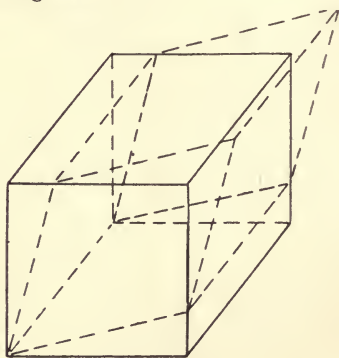


Fig. 55.

Thus a shearing strain is equivalent to a tensile strain in one direction, accompanied by a compressive strain in the perpendicular direction, these strains being each half the shear strain. Fig. 55 shows distortion due to shearing stresses.

In last article the strain in either direction has been shown to be

$$\frac{f}{E} \left( 1 + \frac{1}{m} \right);$$

therefore

$$\phi = \frac{2f}{E} \left( 1 + \frac{1}{m} \right).$$

But

$$\phi = \frac{q}{C} = \frac{f}{C},$$

since the shearing stress and normal stresses to which they are equivalent are equal;

therefore

$$C = \frac{f}{\phi} = \frac{1}{2} \frac{mE}{m+1},$$

and

$$E = 2C \left( 1 + \frac{1}{m} \right).$$

Taking  $m = 4$  for metals,

$$C = \frac{2}{5} E.$$

## 24. Modulus of Elasticity of Volume, or Bulk Modulus.

If an isotropic body be acted on by a stress uniform and normal to its surface at every point, its volume will be increased or diminished.

Let  $V$  be the original volume of the body, and  $v$  the change of volume produced by a stress  $f$  per unit of area, then the cubic strain is

$\frac{v}{V}$ , and

$$f = K \frac{v}{V} \dots\dots\dots(1),$$

where  $K$  is called the *modulus of cubic elasticity* or *bulk modulus*.

Consider a cube of side  $a$  acted on by a uniform stress of intensity  $f$  on all its faces (Fig. 56).

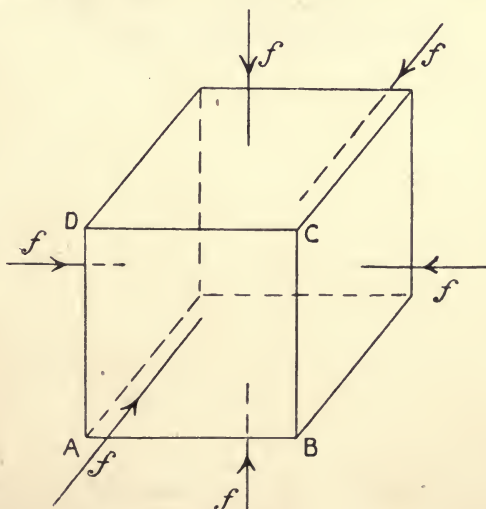


Fig. 56.

The linear strain is  $\frac{1}{3}$  of the cubic strain. For if the volume  $V$  becomes  $(V-v)$ , and the side  $a$  becomes  $(a-x)$ ,

$$(V-v) = (a-x)^3 = a^3 - 3a^2x;$$

therefore

$$3a^2x = v;$$

or

$$\frac{x}{a} = \frac{1}{3} \frac{v}{V}.$$

Let the linear strain  $\frac{x}{a} = \lambda = \frac{f}{E}$ ; then the linear strain being composed of one direct and two lateral strains is equal to

$$\frac{f}{E} \left(1 - \frac{2}{m}\right).$$

$$\text{Hence the cubic strain} = \frac{3f}{E} \left(1 - \frac{2}{m}\right).$$

But the cubic strain is by definition  $= \frac{f}{K}$ ;

therefore

$$\frac{1}{K} = \frac{3}{E} \left(1 - \frac{2}{m}\right),$$

and

$$K = \frac{Em}{3m-6}, \quad \text{If } m = 4, K = \frac{2}{3} E$$

but

$$E = 2C \left(1 + \frac{1}{m}\right).$$

Eliminating  $E$  we get

$$m = \frac{6K + 2C}{3K - 2C},$$

or

$$\frac{1}{m} = \frac{\text{lateral strain}}{\text{longitudinal strain}} = \frac{3K - 2C}{6K + 2C}.$$

*Thus get m experimentally*

## CHAPTER III.

### STRESS-STRAIN DIAGRAMS.—WORKING STRESS.— RESILIENCE.

#### 25. Ultimate, Elastic, and Working Strength.

If a bar of material of uniform section is placed in the testing machine and the load gradually increased, the stresses due to different loads being plotted as ordinates, and the corresponding strains as abscissæ, we get a curve representing the relation of stress and strain.

In the case of wrought iron and mild steel the stress-strain diagram takes the form shown in Fig. 57.

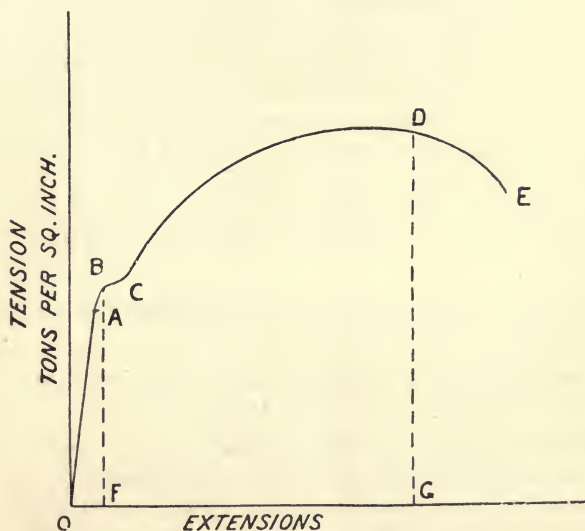


Fig. 57.

*1st stage. The elastic stage.* In this stage from *O* to *A*, the line *OA* is perfectly straight, the strains are *proportional* to the stresses, and the strains are very small. If at any time during this stage the

stress be removed the bar returns to its original length. Between *A* and *B* there is a slight curvature in the diagram, the proportionality between stress and strain ceases, the strain increasing faster than the stress, but the strain still continues very small. If during this time the stress is removed the bar will not entirely return to its original length; there will be a small amount of *permanent set*. At *B*, the *yield point*, the character of the material changes. There is a sudden rapid extension *BC*, without increase of the load, which is very much greater than the previous extension. After the yield point the plastic stage is reached, and the strain increases much faster than the stress until fracture occurs. The actual extension during this stage depends partly on the time during which the load acts; the extension increases with time even when the load is not increased.

In tension tests a maximum stress is reached at some point *D*, after which the extension continues with a reduced load. The specimen breaks at *E* and the portion *DE* of the curve bends back.

In commercial testing the yield point is taken as the elastic limit, and the ordinate *BF* represents this stress limit in tons per square inch.

## 26. Ultimate or Breaking Stress.

When a ductile material such as wrought iron or steel is subjected to a tension test we find that up to the yield point the alteration in the length and cross-section of the specimen is small, but after the yield point, when the loads become large, the section decreases uniformly over the length. When the maximum load is reached, and just before the piece breaks, there occurs a large extension near the place of rupture, and the section there becomes contracted as in Fig. 58. It finally breaks with a load less than the maximum.

*The breaking stress is the maximum load divided by the original sectional area of the piece.*

The breaking stress per square inch is represented in diagram (Fig. 57), by the ordinate *DG*.

This breaking stress is got from a steadily-applied load. Experiments by Wöhler and others show that pieces which are subjected to a continually-varying stress break with a stress of from one-third up to the full breaking stress for a steady pull, according to the amount of variation. These experiments were carried out by Wöhler to determine the effects of repeated alternations of stress from tension to compression, or between high and low values of the same kind of stress. The breaking stress for an indefinite number of



Fig. 58.

repetitions depends on the range through which the stress is varied. It is lowered as the range is increased.

## 27. Plasticity.

When a structure is liable to live loads and shocks it is important not only to ascertain the breaking strength of the material used, but also to determine its power of resisting deformation without rupture. The plasticity of a material is measured by its final elongation and contraction of area.

In wrought iron and mild steel plasticity is combined with a high tensile strength, but this property is absent in cast iron and hard steel.

In specifying for wrought iron and steel it is usual to require a certain percentage of elongation and contraction of area, as well as a certain breaking stress.

The elongation is usually taken on a length of 10 inches.

## 28. Example of tension test.

Specimen of mild steel.

*Original dimensions.* Length, 10 inches ; diameter, 0·897 inches ; area, 0·632 inches.

*Final dimensions.* Length, 12·658 inches ; diameter, 0·584 inches ; area, 0·2678 square inches.

Load	Extension	Remarks
Tons	Inches	
1	0·0011	
2	0·0022	
3	0·0031	
4	0·0042	
5	0·0055	
6	0·0066	
7	0·0086	
8	0·0098	
9	0·0100	
10	0·0110	
11	0·0121	
12	0·0132	
13	0·0144	
14	0·0270	Yield point.
19·86	—	Maximum load.
17·25	—	Breaking load.

### *Elastic limit :*

The elastic limit is evidently reached for a *load* between 13 and 14 tons. The load is taken as 13·5 tons.

*Elastic limit in tons per square inch*

$$= \frac{\text{load at elastic limit}}{\text{original area}}$$

$$= \frac{13.5}{0.632} = 21.43 \text{ tons per square inch.}$$

*Ultimate or breaking stress*

$$= \frac{\text{maximum load}}{\text{original area}}$$

$$= \frac{19.86}{0.632} = 31.52 \text{ tons per square inch.}$$

*Percentage extension on 10 inches*

$$= \frac{\text{final length} - \text{original length}}{\text{original length}} \times 100$$

$$= \frac{12.658 - 10}{10} \times 100$$

$$= 26.58 \text{ per cent.}$$

*Contraction of area per cent.*

$$= \frac{\text{original area} - \text{final area}}{\text{original area}} \times 100$$

$$= \frac{0.632 - 0.2678}{0.632} \times 100$$

$$= 57.62 \text{ per cent.}$$

*Modulus of elasticity :*

Taking the extensions for loads up to 12 tons, the mean extension per ton of load is

$$0.001 \text{ inches.}$$

Now Young's modulus of elasticity

$$E = \frac{fL}{l},$$

where  $f$  is the stress producing the elongation  $l$ , and  $L$  the original length ;

$$f = \frac{1}{0.632} ; \quad l = 0.001 \text{ inches} ; \quad L = 10 \text{ inches.}$$

Therefore

$$E = \frac{10}{0.632 \times 0.001}$$

$$= 15824 \text{ tons per square inch.}$$

## 29. Diagram for mild steel.

Fig. 59 shows a complete diagram of the test. The loads are plotted vertically to a scale of 0.2 inch per ton, and the extensions are plotted horizontally, full size.

The first portion of the diagram is straight up to 13 tons. At 14 tons load we get the very sudden and increased extension which

marks the *yield point* ; after which the curve continues more or less uniformly. The maximum load was 19·86 tons, from which point the curve falls back to 17·25 tons, the load at which the specimen broke. The total extension measured after fracture was 2·658 inches.

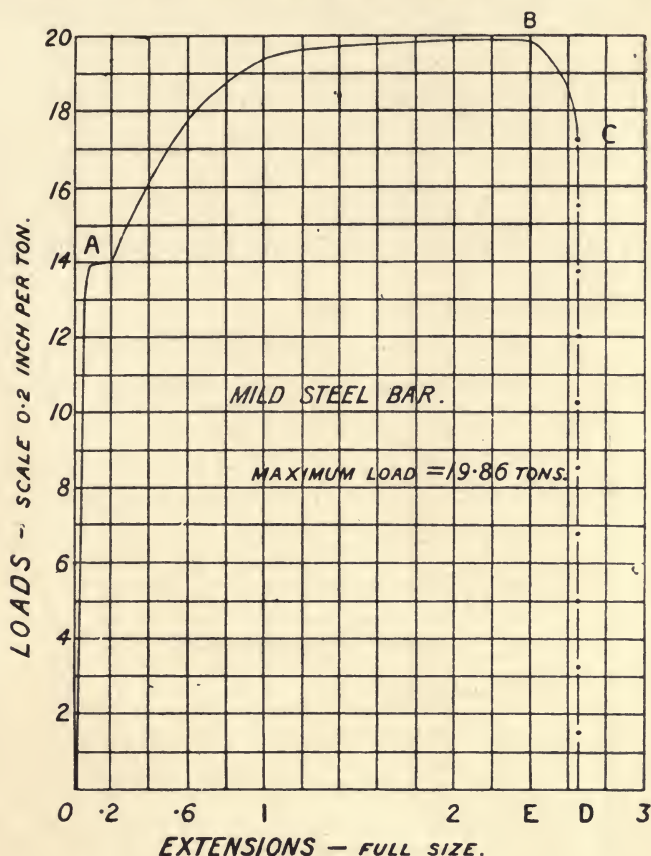


Fig. 59.

### 30. Tension test and diagram for a wrought-iron bar.

*Original dimensions.* Length, 10 inches ; diameter, 0·880 inches ; area, 0·608 square inches.

*Final dimensions.* Length, 12·29 inches ; diameter, 0·742 inches ; area, 0·433 square inches.

*Load at elastic limit,* 10·33 tons.

*Maximum load,* 15·05 tons.

$$\begin{aligned} \text{Stress at elastic limit} &= \frac{10.33}{0.608} \\ &= 17 \text{ tons per square inch.} \end{aligned}$$

*Ultimate or breaking stress*

$$\begin{aligned}
 &= \frac{15.05}{0.608} \\
 &= 24.7 \text{ tons per square inch.}
 \end{aligned}$$

*Percentage elongation on 10 inches*

$$\begin{aligned}
 &= \frac{2.29 \times 100}{10} \\
 &= 22.9 \text{ per cent.}
 \end{aligned}$$

*Percentage contraction of area*

$$\begin{aligned}
 &= \frac{0.608 - 0.433}{0.608} \times 100 \\
 &= 28.9 \text{ per cent.}
 \end{aligned}$$

Fig. 60 gives the diagram, drawn to the same scales as the diagram for mild steel (Fig. 59).

The total extension was 2.29 inches.

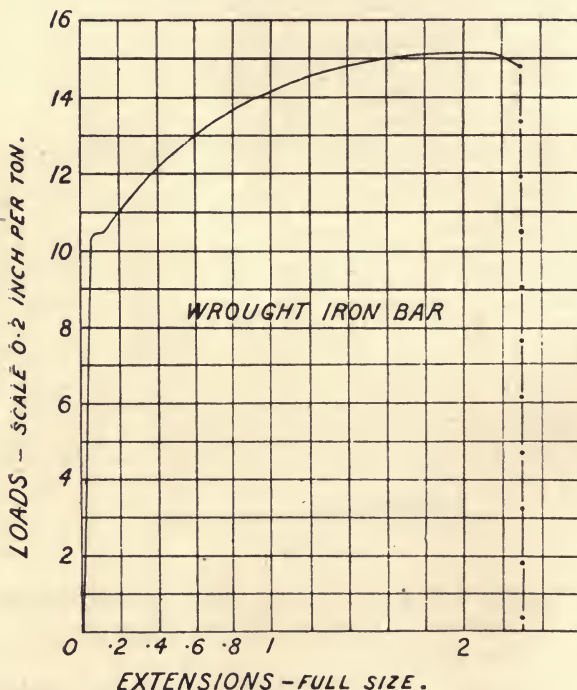


Fig. 60.

31. Work done in fracturing a test bar got from the diagram.

For explanation, reference is made to the diagram for a mild steel bar (Fig. 59).

Work is the product of a force into the linear distance through which it acts. Ordinates on the diagram represent the loads in tons, and the extensions are measured as abscissæ. Draw  $BE$  and  $CD$  verticals. Then the area  $OABCD$  of the diagram represents, to some scale, the work done upon the bar in breaking it. If  $m$  inches = 1 ton, and  $n$  inches = 1 inch of extension, then

$$\text{Work done in inch tons} = \frac{\text{area of diagram in square inches}}{mn}.$$

If we divide this by the volume of the bar we get the work done per cubic inch.

The area of the diagram can be found by a planimeter. In the case of *mild steel bar* (Fig. 59) this area is 10 square inches;  $m = 0.2$ , and  $n = 1$ .

Therefore

$$\text{Work done} = \frac{10}{0.2 \times 1} = 50 \text{ inch tons.}$$

*Work done per cubic inch is*

$$\frac{50}{0.632 \times 10} = \underline{\underline{7.9 \text{ inch tons.}}}$$

In the case of *wrought-iron bar* (Fig. 60) the area of diagram is 6.3 square inches;  $m = 0.2$ ;  $n = 1$ .

Therefore

$$\text{Work done} = \frac{6.3}{0.2 \times 1} = 31.5 \text{ inch tons.}$$

*Work done per cubic inch is*

$$\frac{31.5}{0.608 \times 10} = \underline{\underline{5.18 \text{ inch tons.}}}$$

In order to compare results for the work done *per cubic inch of material* in breaking a bar, the test specimens must be similar, for the ultimate elongation depends for the same material on the original length, and to some extent on the section. Again, the portion  $BC$  of the diagram is seldom satisfactorily determined, and for purposes of comparison it would be better not to include the work represented by  $BCED$ , which is almost entirely work expended in local extension, and to measure only the area  $OABE$ , which represents the work done up to the plastic limit, at which point the bar is for all practical purposes destroyed.

### 32. Compression tests.

If the specimen is very short it will fail by crushing only; if the length is great it will fail by bending or buckling; for intermediate lengths it will fail partly by crushing and partly by bending. In compression tests, which are intended to cause failure by crushing

only, the length of specimens should not be more than  $1\frac{1}{2}$  to 3 times the diameter. In the case of mild steel and other *ductile metals*, as the load is increased the specimen diminishes in length, bulges out laterally and assumes a barrel-shaped form (Fig. 61), and this bulging



Fig. 61.



Fig. 62.

is accompanied by longitudinal cracks. Cast iron fails by sudden rupture on a plane inclined at about  $45^\circ$  with the axis. The most ordinary form of fracture is shearing along an oblique plane making an angle of about  $55^\circ$  with the axis (Fig. 62).

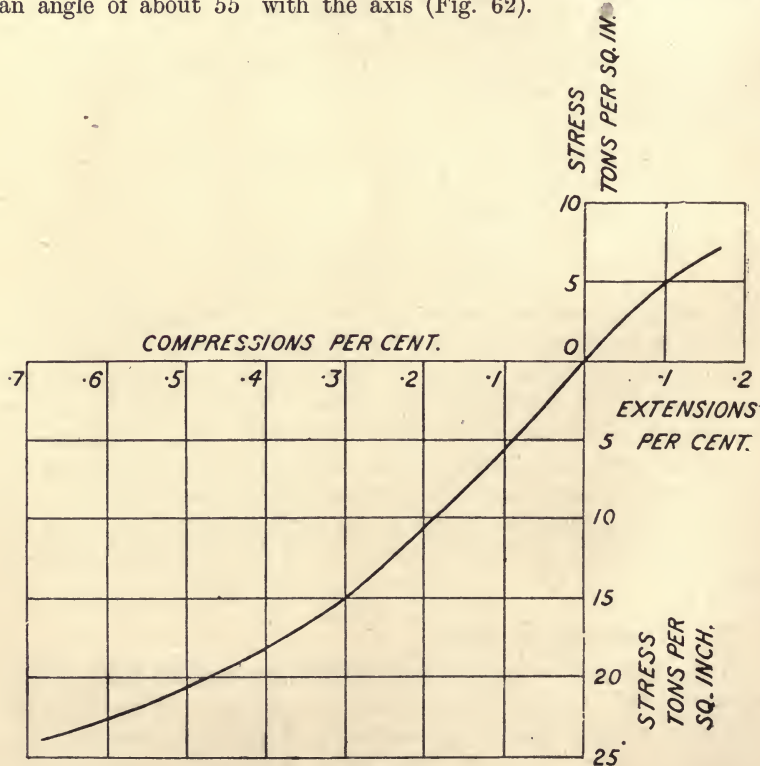


Fig. 63.

**STRESS-STRAIN DIAGRAM FOR CAST IRON.** Fig. 63 gives a diagram for cast iron under tension and compression. It is from experiments by Hodgkinson. Cast metals are practically inelastic and take a set even with small loads. They exhibit very little plasticity. Cast iron is about six times stronger in compression than in tension.

### 33. Torsion Test.

Fig. 64 represents a spiral torsion test for a wrought-iron bar taken in a Buckton's torsion machine. The specimen was 10 inches long,  $\frac{3}{4}$  inch diameter, cross-sectional area 0.442 square inches, and broke with a weight of 166 lbs. at a leverage of 25.04 inches.

The twisting moment at fracture was

$$T = 166 \times 25.04 = 4156.6 \text{ inch lbs.}$$

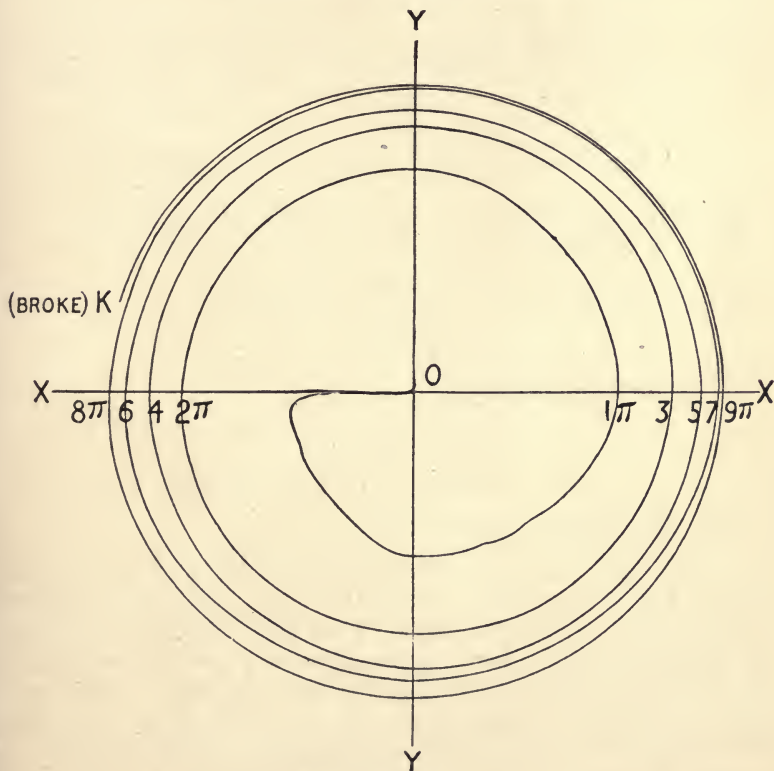


Fig. 64.

**TO FIND THE WORK DONE PER CUBIC INCH.** If the curve, Fig. 64, be developed, we get a diagram, Fig. 65, in which the ordinates represent twisting moments, and the abscissæ angles of twist.

The scale for twisting moments is got by dividing the maximum twisting moment by the length  $OK$  (Fig. 64).

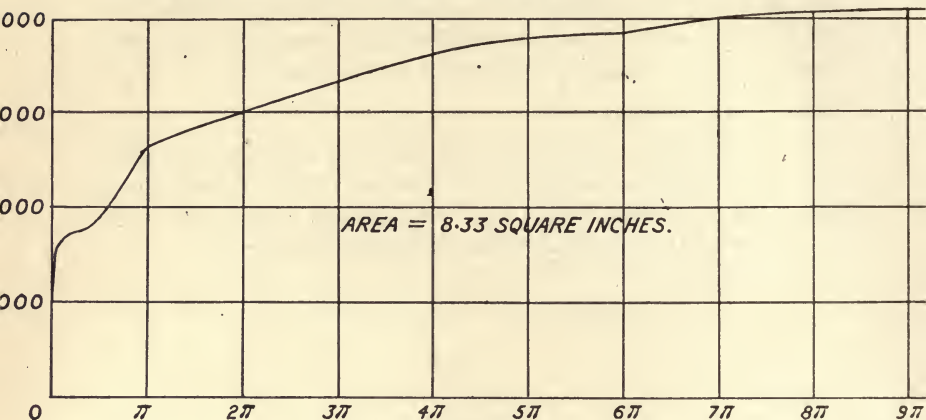


Fig. 65.

$$\text{Vertical scale} = \frac{4156.6}{2.06} = 2000 \text{ in lbs. per inch.}$$

The horizontal scale for twist has been taken as 1 inch =  $2\pi$ .

The area of the diagram, Fig. 65, is 8.33 square inches.

$$\text{Therefore work done} = \frac{8.33 \times 2000 \times 2\pi}{2240} \text{ inch tons}$$

$$= 46.7 \text{ inch tons.}$$

*Work done per cubic inch*

$$= \frac{46.7}{\text{vol.}} = \frac{46.7}{10 \times 0.442}$$

$$= 10.56 \text{ inch tons per cubic inch.}$$

### 34. Dead load and live load.

A dead load is a constant or steady load which does not vary, and produces a constant amount of straining action. Thus the weight of a bridge which includes the weight of the main girders, cross girders, and flooring is a dead load. A live load is a variable load, which alternately comes on to and is removed from the structure, such as a crowd of people or a railway train passing over a bridge. For live loads an allowance is sometimes made for impact or dynamic action.

### 35. Factors of Safety. Working Stress.

In order to allow for possible inaccuracies in determining the loads and stresses, imperfect workmanship, and deterioration of material by exposure, it is necessary that the *working stress*, that is, the maximum stress which a material will bear in actual practice, should be a fraction

only of the breaking strength. The factor by which the breaking strength is divided to get the working stress is called the *factor of safety* :

$$\text{or,} \quad \text{working stress} = \frac{\text{breaking stress}}{\text{factor of safety}}.$$

TABLE OF FACTORS OF SAFETY (UNWIN).

Material	Factors of Safety for			
	A dead Load	A live or varying Load producing		In Structures subject to varying Loads and Shocks
		Stress of one kind only	Equal alternate Stresses of different kinds	
Cast iron ... ..	4	6	10	15
Wrought iron and steel	3	5	8	12
Timber ... ..	7	10	15	20

This method of getting the working stress by dividing the breaking stress by a factor of safety is an empirical one. Wöhler's experiments prove that the safety of a structure depends not on the maximum intensity of stress to which it is exposed, but on the *range of stress* and the number of repetitions of the change of stress. Thus, if a structure is subjected to a steady load, the working stress may be greater than when the structure is subject to a varying stress of one kind (tensile *or* compressive), and when the structure is subjected to alternate stress of opposite kinds (tensile *and* compressive) the working stress must be still less.

The three cases are :

- (a) Steady load.
- (b) Load of one kind applied and removed, many times repeated.
- (c) Tension alternating with compression of the same magnitude, many times repeated.

The range of stress is given as below :

	Max. Stress	Min. Stress	Range of Stress
(a)	$f$	$f$	0
(b)	$f$	0	$f$
(c)	$f$	$-f$	$2f$

And the strength of the material in these three cases is approximately in the ratio of 3 : 2 : 1.

Wöhler considered that the greatest permissible working stresses should be in the ratio of 1 : 2 : 3, according as the members are exposed to tension alternating with compression; to tension alternating with no stress; or to a steady load.

The weakening of material by repeated stresses is called *fatigue*.

Professor Unwin has proposed a formula to include cases of fluctuating stress,

$$f_{\max} = \frac{D}{2} + \sqrt{t(t - nD)},$$

where  $f_{\max}$  is the actual breaking strength when the piece is exposed to stresses varying from  $f_{\max}$  to  $f_{\min}$ , alternately repeated a great number of times.

$D$  is the fluctuation or range of stress;  $t$  the statical breaking stress, and  $n$  a coefficient.

(a) When  $D = 0$  the load is a steady one, and  $f_{\max} = t$ .

(b) When  $D = f_{\max}$  the load alternates with no load, then

$$f_{\max} = 2t(\sqrt{1 + n^2} - n). = 0 = .61 \text{ if } n = \frac{3}{2}$$

(c) When  $D = 2f_{\max}$  the stresses are alternately tensile and compressive of equal intensity. The stress fluctuates from  $f$  to  $-f$ , then

$$f_{\max} = \frac{t}{2n} = .33 t$$

The mean value of  $n = \frac{3}{2}$  for iron and steel, then

$$f_{\max} = \frac{D}{2} + \sqrt{t\left(t - \frac{3}{2}D\right)}.$$

The working stress is  $f_{\max}$ , divided by the factor of safety.

Recent experiments made by Professor Osborne Reynolds and Mr J. H. Smith show "that under a given range of stress the number of reversals before rupture diminishes as the frequency increases, and that hard steels will not sustain more reversals with the same range of stress than mild steels when the frequency of the reversals is great." In the paper in which they describe these experiments, they give the following concise *résumé* of the important results deduced by Wöhler:

1. That wrought iron and steel will rupture with stresses much *below* the statical breaking stress, if such stress be repeated a sufficient number of times.

2. That within certain limits, the *range* of stress, and not the maximum stress, determines the number of reversals necessary for rupture.

3. That as the range of stress is diminished, the number of repetitions for rupture increases.

4. That there is a *limiting range* of stress for which the number of repetitions of stress for rupture becomes infinite.

5. That this limiting range of stress diminishes as the maximum stress increases.

LAUNHARDT'S AND WEYRAUCH'S FORMULÆ. Let  $t$  be the *statical breaking strength*, under a gradually-applied load;  $u$  the *primitive strength*, that is, the breaking strength under repeated stresses of the *same kind*, the stress varying alternately from  $u$  to 0;  $s$  the *vibration strength*, that is, the breaking strength under many times repeated alternating stresses of equal intensities, but *opposite* in sign (tension and compression), that is, the stress varies alternately from  $s$  to  $-s$ .

(A) Suppose a bar of unit cross-sectional area is exposed to stresses ( $f$ ) of the same kind, which vary from a maximum  $f_1$  to a minimum  $f_2$ .

Then the range of stress

$$D = f_1 - f_2 \dots\dots\dots(1).$$

Let

$$\frac{f_2}{f_1} = \frac{\text{min. } f}{\text{max. } f} = Q.$$

By Wöhler's law

$$f_1 \propto D = K \cdot D \dots\dots\dots(2)$$

where  $K$  is an unknown coefficient.

From (1) and (2),

if  $D = 0$ , then  $f_1 = f_2 = t$  and  $K = \infty$ ;

if  $D = u$ , then  $\begin{cases} f_1 = D = u, \text{ and } K = 1 \\ f_2 = 0. \end{cases}$

Launhardt assumed

$$K = \frac{t - u}{t - f_1},$$

as agreeing with these conditions, and giving results agreeing approximately with experiment.

Therefore we get from (2)

$$f_1 = \frac{t - u}{t - f} D = \frac{t - u}{t - f_1} (f_1 - f_2).$$

Thus, the breaking strength for a varying stress of the *same sign* many times repeated is

$$f_1 = u \left( 1 + \frac{t - u}{u} \frac{f_2}{f_1} \right) = u \left( 1 + \frac{t - u}{u} Q \right) \dots\dots\dots(3).$$

This is Launhardt's formula for stresses of the *same sign*.

(B) If the bar be subjected to stresses which are alternately different in kind and vary from a numerical maximum  $f_1$  of one sign to  $f_2$  of the opposite sign, then

$$f_1 + f_2 = D.$$

3 Let  $u = .61t$  Then  $f_1 = .61t \left( 1 + .64 \frac{\text{min}}{\text{max}} \right)$  If  $\text{min} = 0, f_1 = .61t$   
If  $\text{min} = \text{max } f_2 = .61t$

Weyrauch found that  $K = \frac{u-s}{2u-s-f_1}$  for stresses of different kinds agreed approximately with experiment.

Hence by Wöhler's law

$$\begin{aligned} f_1 &= K \cdot D = \frac{u-s}{2u-s-f_1} \cdot D \\ &= \frac{u-s}{2u-s-f_1} (f_1 + f_2) \\ &= u \left( 1 - \frac{u-s}{u} \frac{f_2}{f_1} \right) = u \left( 1 - \frac{u-s}{u} Q \right) \dots\dots\dots(4). \end{aligned}$$

This is Weyrauch's formula for stresses of *different* sign.

Experiments on iron and steel give the following approximate values of  $u$ :

For iron, 13·2 tons per square inch ;

For steel, 18       "       "       "

and the ratios  $\frac{t-u}{u} = \frac{u-s}{u} = \frac{1}{2}$ , that is,  $u = \frac{2}{3} t$ .

Thus equations (3) and (4) give the *breaking stress* of a piece exposed to repeated varying stress, as

$$\text{For iron} = 13\cdot2 \left( 1 \pm \frac{Q}{2} \right).$$

$$\text{For steel} = 18 \left( 1 \pm \frac{Q}{2} \right).$$

With a factor of safety of 3 we get

$$\text{Working stress} \begin{cases} \text{For iron} = 4\cdot4 \left( 1 \pm \frac{Q}{2} \right). \\ \text{For steel} = 6 \left( 1 \pm \frac{Q}{2} \right). \end{cases}$$

The + or - sign for  $Q$  must be used according as the stresses are of the same kind or of different kinds.

The sectional area of the bar or member

$$= \frac{\text{the maximum load}}{\text{working stress}}.$$

The working stress for shearing may be taken as  $\frac{4}{5}$ ths of its value for tension.

The following values of working stress may be adopted in tension and compression :

(a) For a dead load only,

Wrought iron,  $6\frac{1}{2}$  tons per square inch.

Mild steel, 9       "       "       "

(b) For a varying load producing tensile *or* compressive stress only,

Wrought iron,  $4\frac{1}{2}$  tons per square inch.

Mild steel, 6       "       "       "

- (c) For a varying load producing equal alternating stresses in tension and compression,

Wrought iron,  $2\frac{1}{4}$  tons per square inch.

Mild steel, 3 " " "

In the case of cast iron in which the compressive strength is much greater than the tensile strength :

Case (a)	{ Tension,	$1\frac{3}{4}$ tons per square inch.			
	{ Compression,	$5\frac{1}{2}$ " " "			
Case (b)	{ Tension,	$1\frac{1}{4}$ " " "			
	{ Compression,	4 " " "			
Case (c)	Tension,	$\frac{3}{4}$ " " "			

DYNAMIC METHOD. (This method is more fully treated in *Bridge Construction*, by Prof. Claxton Fidler.)

In Art. 37 it is shown that when a load is suddenly applied to a bar the maximum momentary stress produced is double that of the load.

Now, if a bar or member of a structure is already strained by an initial stress  $P$ , and an additional stress  $F$  be suddenly applied, which produces an elongation or shortening  $l$ , we get a stress-strain diagram as in Fig. 66, in which the energy of  $F + P$  is represented by the area  $ABEH$ , and the work done on the bar is represented by the area  $ABGC$ . As these two areas must be equal, we get

$$GE = ED = F,$$

and the dynamically increased stress

$$= BG$$

$$= BE + EG = \overline{P + F} + F \dots\dots\dots(1)$$

$$= \text{statical stress} + \text{variation in stress};$$

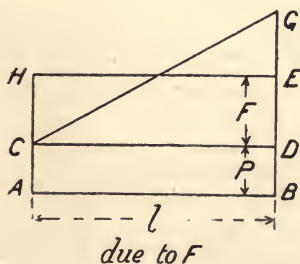


Fig. 66.

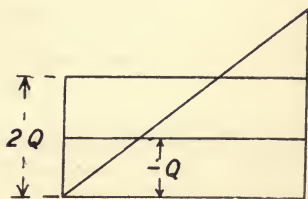


Fig. 67.

or, denoting the initial stress  $DB = P$  by min.  $S$ , and  $BE = \overline{P + F}$  by max.  $S$ , we have  $GE$  the variation in stress = max.  $S$  - min.  $S$ ; that is :—

The maximum dynamically increased stress

$$= \text{max. } S + (\text{max. } S - \text{min. } S).$$

NOTE. If the initial stress is negative ( $-Q$ ), and  $F = 2Q$  (Fig. 67),

The dynamically increased stress, from (1)

$$\begin{aligned}
 &= P + F + F \\
 &= -Q + 2Q + 2Q = 3Q \\
 &= Q + \{Q - (-Q)\} \\
 &= \max. S + \{\max. S - \min. S\},
 \end{aligned}$$

as in this case  $\max. S = Q$ , and  $\min. S = -Q$ .

If, as before,  $t$  = statical breaking stress, and 3 = factor of safety, then the *area of cross-section of member*

$$\begin{aligned}
 &= \frac{\text{dynamically increased stress}}{\frac{t}{3}} \\
 &= \frac{\text{statical stress} + \text{variation in stress}}{\frac{t}{3}} \\
 &= \frac{3}{t} \{\max. S + (\max. S - \min. S)\} \dots\dots\dots (2).
 \end{aligned}$$

In applying this formula, as we shall see later on, the variation of stress is more gradual in the flange members of a girder than in other members, and for these we take *half the variation* only, thus

$$\text{Flange area} = \frac{3}{t} \{\max. S + \frac{1}{2} (\max. S - \min. S)\}.$$

ANOTHER RULE for determining the maximum stress in any member is to add to the dead load stress, the maximum live load stress multiplied by a coefficient. This coefficient is 2.0 in all cases, except for the upper and lower flanges of triangulated girders, for which a coefficient of 1.5 may be used.

Thus, *cross-sectional area*

$$\begin{aligned}
 &\left. \begin{array}{l} \text{dead load stress} + \frac{2}{1\frac{1}{2}} \\ \text{or} \end{array} \right\} \text{live load stress} \\
 &= \frac{\dots\dots\dots}{\frac{t}{3}} \dots\dots\dots (3).
 \end{aligned}$$

It should be noted that equation (2) gives a rather greater area for the braces near the middle of span where the *variation* in stress is equal to

dead load stress + live load stress

than rule formula (3).

### 36. Strength of wrought iron—steel—cast iron.

The strength of a material depends greatly on the mechanical treatment during manufacture. Material that has been worked during manufacture by rolling and forging is increased in strength. Wrought iron exhibits greater tensile strength when tested in the direction of rolling than when tested across the direction of rolling.

Mild steel contains from .1 to .25 per cent. of carbon, and hard steel from .25 to 1.4 per cent. of carbon. Mild steel, on account of its greater tensile strength and plasticity, has practically superseded wrought iron for constructional purposes.

The tensile strength and shearing strength of steel increase as the percentage of carbon increases, but the plasticity, as measured by the elongation and reduction of area, diminishes.

Cast iron is inelastic; there is practically no proportionality between stress and strain, and consequently no elastic limit.

The properties of each of these three metals are shown in the following statement :

**WROUGHT IRON BARS.** Tensile strength, 21 to 24 tons per sq. inch.

„ plates with grain, „ 20 to 22 „ „

„ plates across grain, „ 18 to 20 „ „

Extension on an 8-inch length, 20 to 25 per cent.

Contraction of area, 40 to 45 per cent.

Crushing strength, 16 to 20 tons per square inch.

Shearing strength, 16 to 20 tons per square inch.

$E$  = Young's modulus of elasticity, 12,500 tons per square inch.

$C$  = modulus of rigidity, 5,000 tons per square inch.

**MILD STEEL** (about 0.2 per cent. of carbon) :

Tensile strength, 28 to 30 tons per square inch.

Extension on an 8-inch length, 25 per cent.

Contraction of area, 45 to 55 per cent.

Shearing strength, 22 to 24 tons per square inch.

$E$  = Young's modulus of elasticity, 13,000 to 13,500 tons per square inch.

$C$  = modulus of rigidity, 5,200 to 5,500 tons per square inch.

**CAST IRON.** Tensile strength, 8 to 12 tons per square inch.

Crushing strength, 40 to 50 „ „

Shearing strength, 6 to 12 „ „

$E$  = Young's modulus, 5,000 to 6,000 tons per square inch.

$C$  = modulus of rigidity, 2,500 to 4,000 tons per square inch.

### 37. Work done in extending or compressing a bar within the elastic limit. Resilience.

Let  $L$  be the length of the bar,  $l$  the elongation or compression,  $A$  the area of cross-section.

#### CASE I. GRADUALLY-APPLIED LOAD.

Let the load be *gradually* applied, increasing from 0 up to  $F$ ; its mean value is  $\frac{F}{2}$ , and the *work done* is therefore  $\frac{F}{2}l$ , represented by the area  $OAB$  (Fig. 68).

But  $F = fA = EA \frac{l}{L}$ ,  $f$  being the intensity of stress.

Therefore

$$\text{Work done} = \frac{E}{2} A \frac{l^2}{L} = \frac{1}{E} \left( \frac{F}{A} \right)^2 \frac{AL}{2} = \frac{1}{2} f^2 \frac{AL}{E}.$$

$$\text{Work done per unit of volume} = \frac{f^2}{2E}.$$

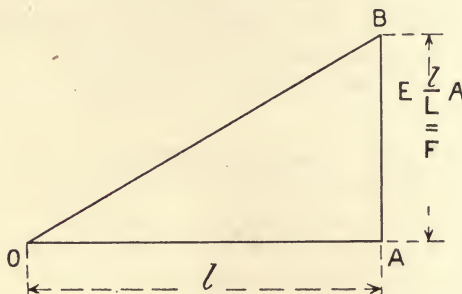


Fig. 68.

As the material is elastic, this measures also the strain energy stored in the piece.

If  $f$  = stress at the elastic limit, then the work done is called the *resilience* of the bar.

*Resilience* is therefore the work done in deforming a piece up to the elastic limit, or it may be defined as the energy stored up in the piece in consequence of a strain up to the elastic limit.

#### CASE II. LOAD SUDDENLY APPLIED WITHOUT VELOCITY.

If a load  $F$  be *suddenly* applied without velocity to a bar of length  $L$ , let  $l$  be the deformation (Fig. 69). Then the work done by the external load must be equal to the energy stored in the bar.

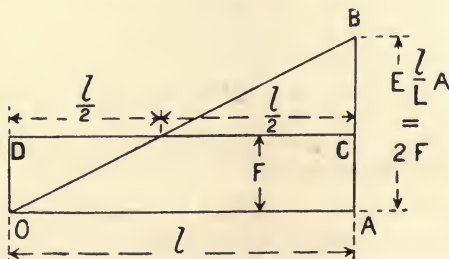


Fig. 69.

Let  $T$  be the maximum stress produced in the bar. Its initial value is nil, so the mean resistance is  $\frac{T}{2}$ .

The work done on bar  $= \frac{T}{2} l = \frac{1}{2} \frac{EA}{L} l^2 = \text{area } OAB.$

Work done by external load  $= Fl = \text{area } OACD.$

Therefore  $\frac{Tl}{2} = \frac{1}{2} \frac{EA}{L} l^2 = Fl,$

and  $T = 2F$ ; or  $E \frac{l}{L} = 2 \frac{F}{A}.$

*Hence the maximum intensity of the stress caused by the sudden application of a constant load, without initial velocity, is double the intensity of the load itself.*

### CASE III. SUDDENLY-APPLIED LOAD WITH VELOCITY.

Suppose a weight of  $W$  lbs. dropped from a height of  $h$  inches and stopped by a bar (Fig. 70).

Let  $T$  be the total stress produced, and  $l$  the elongation or shortening.

The work done by the falling weight  $= W(h + l)$  inch-lbs., and this must be equal to the energy stored.

Therefore

$$\frac{T}{2} l = \frac{1}{2} \frac{EA}{L} l^2 = W(h + l) = Wh \text{ if } l \text{ is very small.}$$

If  $V$  = velocity of  $W$  at the moment of impact

$$\frac{WV^2}{2g} = \frac{Tl}{2} = W(h + l).$$

*Example 1.* What should be the area of cross-section of a wrought-iron bar 20 feet long in order that it may resist the energy of a load of  $\frac{1}{2}$  a ton falling through a height of 3 inches, the resultant load being along the axis?  $E = 13,000$  tons per square inch; elastic limit,  $9\frac{1}{2}$  tons per square inch.

$$W(h + l) = \frac{T}{2} l = \frac{1}{2} fAl \dots\dots\dots(1),$$

where  $f = 9.5$  tons per square inch,  $W = \frac{1}{2}$  ton,  $h = 3$  inches  $= 0.25$  foot,  $l$  = deformation at the elastic limit.

$$\text{Now} \quad 9.5 = E \frac{l}{L} = 13000 \frac{l}{20}.$$

$$\text{Therefore} \quad l \text{ (in feet)} = \frac{20 \times 9.5}{13000} = 0.0146,$$

and from (1)

$$A = \frac{2 \times \frac{1}{2} (0.25 + 0.0146)}{9.5 \times 0.0146} = \frac{0.2646}{0.1387} = 1.908 \text{ sq. inches.}$$

Thus a load having an *intensity* of only  $\frac{0.5 \text{ ton}}{1.9} = 0.26$  ton per

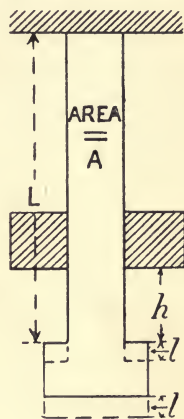


Fig. 70.

square inch suddenly applied to this bar, with a velocity due to its fall from a height of only 3 inches, produces momentarily the same maximum intensity of stress as a gradually-applied load of  $9\frac{1}{2}$  tons per square inch.

*Example 2.* What would be the elongation and consequent intensity of stress produced in a wrought-iron bar 25 feet long and  $1\frac{1}{4}$  inches diameter when extended longitudinally by a weight of 13 cwt. falling through a height of 7 inches?  $E = 12,500$  tons per square inch.

Formula is  $W(h + l) = \frac{1}{2} \frac{EA}{L} l^2$ .

If we neglect  $Wl$  as being comparatively small, we get

$$Wh = \frac{1}{2} \frac{EA}{L} l^2 \dots\dots\dots(1),$$

where  $W = 13$  cwt. =  $0.65$  ton,  $h = 7$  inches =  $0.583$  feet,  $L = 25$  feet,  $E = 12,500$  tons per square inch,  $A = 1.25^2 \times 0.7854 = 1.2272$  square inches.

Then

$$\begin{aligned} l &= \sqrt{\frac{2 \times 25 \times 0.65 \times \frac{7}{12}}{12500 \times 1.2272}} = \sqrt{\frac{4.55}{3681.6}} \\ &= \sqrt{0.001236} = 0.0352 \text{ feet.} \end{aligned}$$

Intensity of stress

$$= E \frac{l}{L} = 12500 \times \frac{0.0352}{25} = 17.6 \text{ tons per square inch.}$$

If the load were gradually applied the intensity of stress would be only  $\frac{0.65}{1.2272} = 0.53$  ton per square inch.

## CHAPTER IV.

### COMPOUND STRESSES.

**38. Combination of a pair of simple longitudinal stresses in directions at right angles to one another. Ellipse of stress.**

In simple tension and compression the *direction* of the stress is the same for all planes, but the *intensity* varies.

Consider a small rectangular block  $ABCD$  of material of unit thickness whose sides are perpendicular to the stresses  $f_1, f_2$ , both of the same sign (Fig. 71).

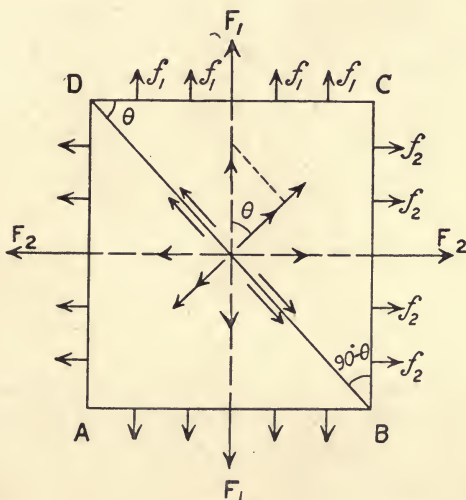


Fig. 71.

The total stress on faces  $AB$  and  $CD = f_1 AB$ .

The total stress on faces  $BC$  and  $AD = f_2 BC$ .

Consider any inclined plane  $BD$ , the normal to which makes an angle  $\theta$  with the direction of  $f_1$ .

Let  $f_n$  and  $f_t$  be the intensities of the normal and tangential stresses on plane  $BD$ .

Resolving perpendicular to  $BD$  for the normal stress

$$f_n BD = f_1 AB \cos \theta + f_2 BC \sin \theta.$$

Therefore  $f_n = f_1 \cos^2 \theta + f_2 \sin^2 \theta$  .....(1).

To get the tangential stress resolve along  $BD$

$$f_t BD = f_1 AB \sin \theta - f_2 BC \cos \theta.$$

Therefore  $f_t = (f_1 - f_2) \sin \theta \cos \theta$  .....(2).

The tangential stress is a maximum when  $\theta = 45^\circ$ .

$$\text{Max. } f_t = \frac{f_1 - f_2}{2}.$$

The normal stress on the same surface inclined at  $45^\circ$  is

$$\frac{f_1 + f_2}{2}.$$

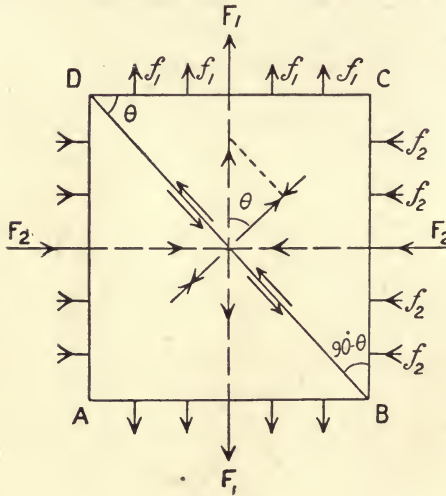


Fig. 72.

If the stresses  $f_1$  and  $f_2$  are of opposite sign,  $f_1$  being a tension or pull, and  $f_2$  a thrust (Fig. 72), then the normal stress on  $BD$

$$f_n = f_1 \cos^2 \theta - f_2 \sin^2 \theta,$$

and the tangential stress on  $BD$

$$f_t = (f_1 + f_2) \sin \theta \cos \theta = \frac{f_1 + f_2}{2} \sin 2\theta.$$

If the two stresses are of *equal intensity*  $f_1$  and of *opposite sign*, then on a plane *inclined at*  $45^\circ$  there is no normal stress. There exists only a tangential or shearing stress on the two planes at  $45^\circ$  to the axes along which the stresses act. The intensity of this shearing stress is  $\frac{2f_1}{2} = f_1$ .

39. To find the magnitude and direction of the resultant stress on the plane BD (Fig. 71).

Let  $f$  be the resultant stress. Compounding equations (1) and (2),

$$\begin{aligned} f \cdot BD &= \sqrt{f_1^2 AB^2 + f_2^2 BC^2} \\ &= BD \sqrt{f_1^2 \cos^2 \theta + f_2^2 \sin^2 \theta}. \end{aligned}$$

Therefore  $f = \sqrt{f_1^2 \cos^2 \theta + f_2^2 \sin^2 \theta} = \sqrt{f_n^2 + f_t^2} \dots\dots\dots (3).$

Let  $\alpha$  be the angle which resultant makes with the direction of  $f_1$ , then

$$\tan \alpha = \frac{f_2 BC}{f_1 AB} = \frac{f_2 \sin \theta}{f_1 \cos \theta} = \frac{f_2}{f_1} \tan \theta \dots\dots\dots(4).$$

40. Ellipse of stress.

The resultant stress may be found graphically as follows (Fig. 73) :

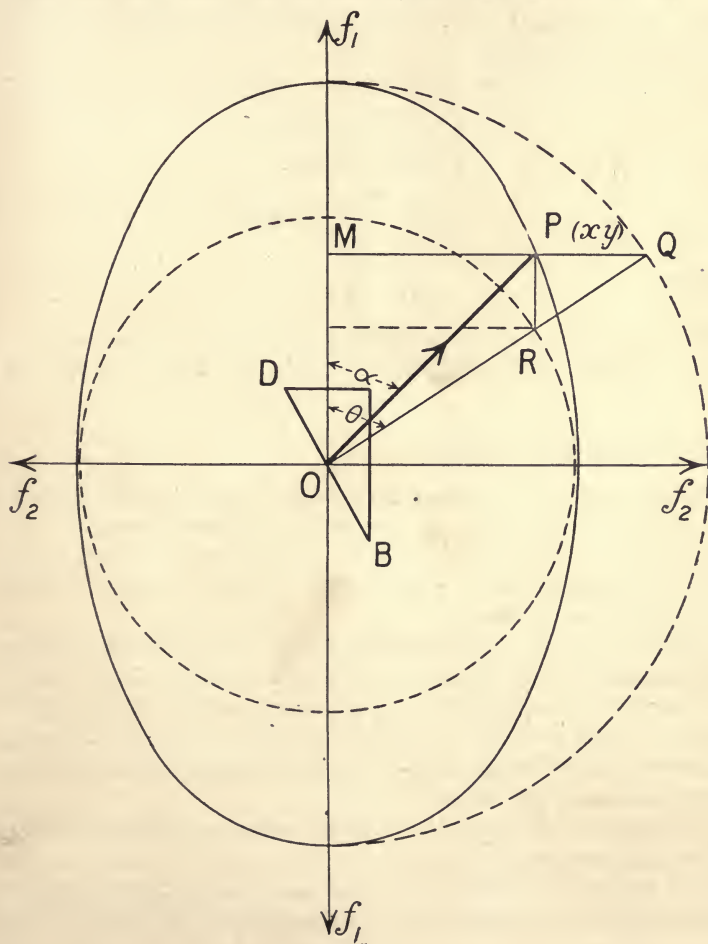


Fig. 73.

On the perpendicular to the plane  $BD$  set off  $OQ$  to represent  $f_1$ , and  $OR$  to represent  $f_2$ . Draw  $QM$  perpendicular to  $f_1$ , and  $RP$  parallel to  $f_1$  to meet  $QM$  in  $P$ . Describe circles with radii  $f_1$  and  $f_2$  from  $O$  as centre.

Then  $OP$  represents the resultant stress on the plane  $BD$  in magnitude and direction, and the locus of  $P$  is an ellipse.

$$\text{Now} \quad OQ = f_1; \quad OR = f_2,$$

$$\text{and} \quad OM = OQ \cos \theta = f_1 \cos \theta,$$

$$PM = OR \sin \theta = f_2 \sin \theta.$$

$$\text{Therefore} \quad OP = \sqrt{f_1^2 \cos^2 \theta + f_2^2 \sin^2 \theta}$$

$$\text{and} \quad \tan \alpha = \frac{PM}{OM} = \frac{f_2}{f_1} \tan \theta.$$

Hence from equations (1) and (2)  $OP$  is the resultant stress in magnitude and direction.

Let  $x, y$  be coordinates of  $P$ ,

$$x = PM = f_2 \sin \theta,$$

$$y = OM = f_1 \cos \theta,$$

$$\text{and} \quad \frac{x}{f_2} = \sin \theta, \quad \frac{y}{f_1} = \cos \theta;$$

$$\text{therefore} \quad \frac{x^2}{f_2^2} + \frac{y^2}{f_1^2} = 1.$$

Thus  $P$  lies on an ellipse of which  $f_1$  and  $f_2$  are the semi-axes.

If  $f_1 = f_2$ ,  $OP$  is at right angles to  $BD$ , and the ellipse becomes a circle.

#### 41. Principal stresses.

Planes on which the stresses are *wholly normal* are called *planes of principal stress*, and the stresses themselves *principal stresses*.

A state of stress in two dimensions can always be represented by an ellipse, the semi-axes of which are the principal stresses, and their directions the axes of stress.

Suppose  $f_1$  and  $f_2$  of the preceding article to be replaced by stresses of any magnitude and direction on two faces at right angles. Resolve these stresses into normal and tangential components. The tangential components must be equal. Let  $f_n$  and  $f_n'$  be the intensities of normal components, and  $q$  the intensity of the equal tangential components on the two planes at right angles (Fig. 74).

To determine the planes of principal stress, and the magnitude of the principal stress on these planes.

It is required to find a plane  $DB$  such that the stress on it is *wholly normal*, and to determine  $f$ , the intensity of that stress. Let  $\theta$

be the angle which  $BD$  makes with  $DC$ . Consider the equilibrium of the right prism  $DBC$  of unit thickness.

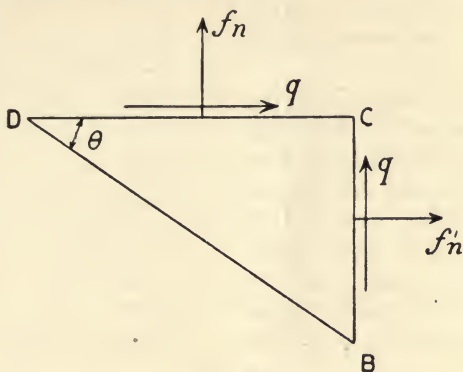


Fig. 74.

Resolve parallel to  $BC$

$$f \cdot DB \cdot \cos \theta = qBC + f_n DC,$$

or

$$f - f_n = q \tan \theta \dots\dots\dots(1).$$

Similarly, resolving parallel to  $DC$ ,

$$f - f'_n = q \cot \theta \dots\dots\dots(2).$$

Subtracting,

$$f_n - f'_n = q (\cot \theta - \tan \theta) = 2q \cot 2\theta,$$

$$\tan 2\theta = \frac{2q}{f_n - f'_n} \dots\dots\dots(3).$$

Two values of  $\theta$  satisfy this equation, that is  $\theta$  and  $90^\circ + \theta$ .

Thus there are two planes at right angles to each other, on which the stress is wholly normal. The value of the principal stress on these planes is got by multiplying (1) and (2),

$$(f - f_n)(f - f'_n) = q^2.$$

The roots of this quadratic are the stresses required.

Let  $AB$  and  $CD$  (Fig. 75) be the pair of rectangular planes through  $O$  upon which the stresses are wholly normal; they are the *planes of principal stress*; the stresses themselves are called the *principal stresses* at  $O$ , and the axes  $OX$  and  $OY$  the axes of principal stress at that point.

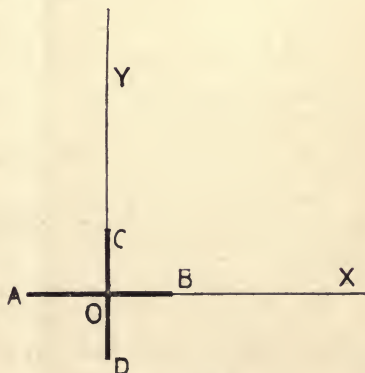


Fig. 75.

## 42. Principal stresses in a beam.

In the case of a loaded beam we get a longitudinal stress combined with shearing stress on longitudinal and transverse planes (Fig. 76).

Consider a small triangular right prism of unit thickness  $BCD$  bounded by a plane  $BD$ , inclined at  $\theta$  to the vertical, the vertical plane  $DC$ , on which there is both a normal and a shearing stress, and the horizontal plane  $BC$ , on which there is only shearing stress. Let  $f_n$  be the intensity of the normal stress on  $DC$ , and let  $q$  be the intensity of the shearing stress, which is the same on  $BC$  and  $CD$  (see Art. 21).

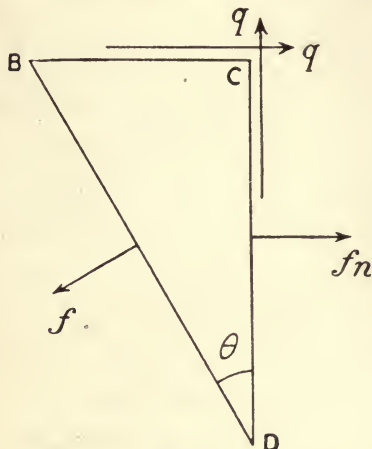


Fig. 76.

To find on what plane  $BD$  (as measured by  $\theta$ ) there is a normal stress only, and to find  $f$  the intensity of that normal stress.

Resolving horizontally,

$$fBD \cos \theta = f_n DC + qBC,$$

$$\text{or} \quad (f - f_n) \cos \theta = q \sin \theta \dots\dots\dots(1).$$

Resolving vertically,  $fBD \sin \theta = qDC$ ,

$$\text{or} \quad f \sin \theta = q \cos \theta \dots\dots\dots(2).$$

From (1) and (2) eliminating  $f$  we get

$$f_n = q (\cot \theta - \tan \theta) = 2q \cot 2\theta.$$

$$\text{That is} \quad \tan 2\theta = \frac{2q}{f_n} \dots\dots\dots(3).$$

Also from (1) and (2)  $f(f - f_n) = q^2$ ,

$$f = \frac{f_n}{2} \pm \sqrt{\frac{f_n^2}{4} + q^2} \dots\dots\dots(4).$$

Two values of  $\theta$  satisfy equation (3), that is  $\theta = \theta$  and  $\theta = 90^\circ + \theta$ . Thus there are two planes of principal stress at right angles to one another.

The positive value  $f_1 = \frac{f_n}{2} + \sqrt{\frac{f_n^2}{4} + q^2}$  is the greatest principal stress of the same kind as  $f_n$ . The lesser principal stress is  $f_2 = \frac{f_n}{2} - \sqrt{\frac{f_n^2}{4} + q^2}$ , which occurs on a plane at right angles to the former.

The *maximum intensity of shearing stress* occurs on planes inclined at  $45^\circ$  to the planes of principal stress, and by Art. 38 its value is

$$\frac{f_1 - f_2}{2} = \sqrt{\frac{f_n^2}{4} + q^2}.$$

If the principal planes (or directions of axes of stress) be drawn by short lines through a series of points taken very close together on the elevation of a beam, then these lines will form two series of curves intersecting each other at right angles, called the *curves or lines of principal stress*, and the tangents to these curves at any point where they intersect are the planes of principal stress (or the axes of stress) at that point.

In the case of a beam supported at both ends the curves convex upwards are curves of compression, and those convex downwards curves of tension.

Again, when  $\theta = 90^\circ$  or  $0^\circ$ , then  $q = 0$ ; and when  $\theta = 45^\circ$ , then  $f_n = 0$ .

Thus the *curves of principal stress* cut the upper and lower surfaces at right angles, and cut the neutral axis at an angle of  $45^\circ$ .

Curves of *maximum shearing stress* cut the upper and lower surfaces at  $45^\circ$ , and touch the neutral axis.

Curves of principal stress are sketched in Fig. 77.

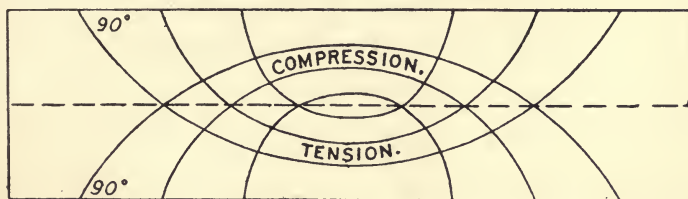


Fig. 77.

### 43. General equations connecting stress and strain.

Suppose an isotropic body, that is, a body having the same elastic properties in all directions, to be acted on by three *principal stresses*  $f_1, f_2, f_3$ , the axes of reference coinciding with the principal axes.

Let the strains be  $\lambda_1, \lambda_2, \lambda_3$  along these axes.

Now, by the Principle of Independence of Stresses, the resultant effect of a compound stress can be found by calculating the system of strain due to each stress taken separately.

Thus  $\lambda_1$  is the sum of the longitudinal strain  $\frac{f_1}{E}$  due to  $f_1$ , and the two lateral strains  $-\frac{f_2}{mE}$  and  $-\frac{f_3}{mE}$  due to  $f_2$  and  $f_3$ .  
 $\frac{1}{m}$  is Poisson's ratio.

Thus  $\lambda_1 = \frac{f_1}{E} - \frac{f_2 + f_3}{mE},$

or  $E\lambda_1 = f_1 - \frac{f_2 + f_3}{m}.$

Similarly  $E\lambda_2 = f_2 - \frac{f_1 + f_3}{m},$

$E\lambda_3 = f_3 - \frac{f_1 + f_2}{m}.$

As a special case, suppose a bar pulled or pushed along the axis of  $f_1$ , left quite free along the axis of  $f_2$ , and all lateral strain prevented along the axis of  $f_3$ .

Then  $f_2 = 0,$  and  $\lambda_3 = 0.$

Thus  $E\lambda_1 = f_1 - \frac{f_3}{m} \dots\dots\dots(1),$

$E\lambda_2 = -\frac{f_1 + f_3}{m},$

$0 = f_3 - \frac{f_1}{m} \dots\dots\dots(3).$

From (3)  $f_3 = \frac{f_1}{m}.$

And from (1)  $E\lambda_1 = f_1 - \frac{f_1}{m^2} = f_1 \left(1 - \frac{1}{m^2}\right),$

or  $\frac{f_1}{\lambda_1} = E \frac{m}{m^2 - 1}.$

STRENGTHS AND WEIGHTS OF MATERIALS.

Material	Weight in Lbs. per Cubic Foot	Ultimate Strength. Tons per Square Inch			Modulus of Elasticity <i>E</i> .	Modulus of Rigidity <i>C</i> .
		Tension	Crushing	Shearing		
Wrought-iron bars	480	21 to 24	18	18 to 20	12,500 to 13,000	5,000 to 5,500
Wrought-iron plates, with fibre ... ..	480	20 to 24	18	10	—	—
Wrought-iron plates, across the fibre ...	480	18 to 20	—	16 to 20	—	—
Mild steel ... ..	490	28 to 32	—	22 to 24	13,000 to 13,500	5,500
Cast-iron ... ..	450	8	40 to 50	9 to 12	4,500 to 6,000	2,500 to 4,000
Copper, cast ... ..	550	10	—	—	5,000 to 7,000	2,000 to 2,500
Copper, rolled ... ..	550	14	—	—	5,500 to 7,500	2,100 to 2,900
Brass, cast ... ..	500	8 to 10	5 to 6	—	4,000	2,000
Hemp rope ... ..	—	4	—	—	—	—
Leather belting ... ..	—	2	—	—	—	—
Oak ... ..	43 to 62	4½	3	—	500 to 700	—
Red pine ... ..	35	4 to 6	2½	—	650	—
Yellow pine ... ..	26	2 to 6	2 to 3	—	780	—
Pitch pine ... ..	40 to 45	4	3½	—	—	—
White pine ... ..	25	1½ to 3	2	—	480	—
Ash ... ..	47	4 to 6	4	—	760	—
Teak ... ..	41 to 55	4 to 6	5	—	1,000	—
Elm ... ..	34	6	4	—	600	—
Greenheart ... ..	58 to 72	3	5½	—	750	—
Cement, set 1 week	86 to 94	0·16	7 8	—	—	—

## STRENGTH OF STONES TO RESIST CRUSHING.

Material	Weight in Lbs. per Cubic Foot	Crushing Weight in Lbs. per Square Inch
Brick stocks ... ..	112	2,240
Do. Staffordshire blue	112	6,200
Concrete, ordinary ... ..	120	450 to 750
Do. cement ... ..	135	1,680
Limestone ... ..	130 to 160	7,500 to 9,000
Sandstone ... ..	135	5,700 to 10,000
Slate ... ..	175	10,800
Granite ... ..	170	10,800 to 20,000
Brickwork ... ..	112	600 to 1,200
Masonry ... ..	116 to 144	800

## EXERCISES.

1. A bar of mild steel 10 feet long and 2 inches diameter is stretched  $\frac{1}{50}$ th of an inch by a load of 7 tons acting along the axis of the bar. Find the intensity of stress, the strain, and the modulus of elasticity.

*Ans.* 2·2 tons per square inch ;  $\frac{1}{80000}$  ; 13,200 tons per square inch.

2. A wrought-iron tie-rod  $\frac{3}{4}$  inch thick has to support a load of 20 tons. Determine its width, assuming a working stress of 8,000 lbs. per square inch.

*Ans.* 7·46 inches.

3. A mild-steel tie-rod 26 feet long is made of an angle-iron  $4'' \times 4'' \times \frac{1}{2}''$ . If it is pulled by a force of 20 tons acting along its axis, determine the intensity of tensile stress and the elongation.

*Ans.* 5·3 tons per square inch ; 0·127 inches.

4. What should be the diameter of the stay-bolts of a boiler in which the pressure is 120 lbs. per square inch, allowing one stay-bolt to each square foot of surface, and a stress of 5 tons per square inch of section of the bolts ?

*Ans.*  $1\frac{3}{8}$  inches.

5. A rectangular wooden post 8 inches broad rests on a masonry wall, and is loaded with 6 tons. Determine the necessary width of the post if the working stress of the masonry is 140 lbs. per square inch.

*Ans.* 12 inches.

6. The rod of a hydraulic hoist is 40 feet long and  $1\frac{1}{2}$  inches diameter; it is attached to a plunger 5 inches diameter, working under a pressure of 1,000 lbs. per square inch. Find the alteration in length of the rod.

$E = 30,000,000$  lbs. per square inch.

*Ans.* 0·18 inch.

7. A brick wall 2 feet thick, 10 feet high, weighing 110 lbs. per cubic foot, is supported on timber columns 8 inches square, 10 feet long, placed 12 feet apart centre to centre. Find the compressive stress in column. If the crushing stress of the timber is 5,000 lbs. per square inch, find to what height the wall can be built, allowing a factor of safety of 10.

*Ans.* 412·8 lbs. per square inch ; 12·1 feet.

8. A bar of wrought-iron 20 feet long,  $1\frac{1}{2}$  inches diameter, is heated to  $160^{\circ}\text{C}$ . ; while at this temperature it is made to connect two walls of a house which have fallen outwards from the perpendicular, by means of washers and nuts screwed on at the ends. If the walls do not yield, find the pull exerted on them when the bar has cooled to  $60^{\circ}\text{C}$ .

Coefficient of expansion for wrought-iron = ·0000124 for  $1^{\circ}\text{C}$ .

$E = 30,000,000$  lbs. per square inch.

*Ans.* 65,844 lbs.

9. A wrought-iron bar 4 square inches in sectional area has its ends fixed between two immoveable blocks when the temperature is  $30^{\circ}\text{C}$ . Find the pressure that will be exerted on the blocks when the temperature is  $100^{\circ}\text{C}$ .

Coefficient of expansion = ·0000125.

$E = 30,000,000$  lbs. per square inch.

*Ans.* 105,000 lbs.

10. Determine the resilience of a steel tie-bar,  $1\frac{1}{2}$  inches in diameter and 5 feet long, if the elastic limit is reached under a load of 20 tons.

Modulus of elasticity = 13,000 tons per square inch.

*Ans.* 1164·8 inch lbs.

11. Calculate the resilience in foot lbs. of a bar of wrought-iron 6 inches diameter, 2 feet long.

Elastic limit = 10 tons per square inch.

$E = 13,000$  tons per square inch.

*Ans.* 487·4 foot lbs.

12. Steam at a pressure of 220 lbs. per square inch is suddenly admitted on to a piston at rest 15 inches diameter. If the piston-rod be 6 feet long and 3 inches diameter, find the maximum stress produced in the rod, the amount of compression, and the work done on the rod at the maximum compression.

Modulus of elasticity = 30,000,000 lbs. per square inch.

*Ans.* 10,998 lbs. per square inch ; 0·026 inches ; 84·2 foot lbs.

13. Find the resilience of a bar of steel 10 feet long and half a square inch in sectional area, stress limit of elasticity 50,000 lbs. per square inch.

Modulus of elasticity = 35,000,000 lbs. per square inch.

*Ans.* 178·6 foot lbs.

14. A bar of steel 10 feet long and 3 inches diameter is subjected to a pulling force of 100 tons. If the modulus of elasticity = 13,000 tons per square inch, determine the number of foot lbs. of energy stored in the bar.

*Ans.* 1,220 foot lbs.

15. A round bar of wrought-iron is 30 feet long and  $1\frac{1}{2}$  inches diameter. Find the tensile load which, if suddenly applied, would cause an instantaneous elongation of the bar of 0·1 inch.

*Ans.* 3·07 tons.

16. A waggon weighing 5 tons, attached to a rope, is travelling down a slope at 4 miles an hour, when it is suddenly stopped. If the rope is  $2\frac{1}{4}$  inches diameter, and its length 600 feet at the moment the waggon is stopped, determine the maximum tension in the rope.

$E = 30,000,000$  lbs. per square inch.

*Ans.* 24 tons, nearly.

17. A bar of steel is 10 feet long and 1 inch diameter. Elastic limit 18 tons per square inch. Modulus of elasticity 33,000,000 lbs. per square inch. Determine the greatest weight that can be dropped on to the bar from a height of 12 inches, and the alteration in length of the bar.

*Ans.* 214·5 lbs. ; 0·161 inches.

18. A wrought-iron bar  $\frac{1}{4}$  inch diameter has two marks made on it 20 inches apart, and it is found that in the testing machine this distance is increased by ·00416, ·00832, ·01248, ·01664, ·031 inches, when the pull is 0·125, 0·250, 0·375, 0·500, 0·625 tons. Determine the yield point and modulus of elasticity.

*Ans.* 12·5 tons per square inch ; 12,020 tons per square inch.

19. If the normal intensity of tensile stress on a transverse section of a bar be 4 tons per square inch, determine the tangential stress on a plane inclined to the normal section at an angle of  $30^\circ$  ; find also the intensity of the resultant stress on that section.

*Ans.*  $\sqrt{3}$  tons per square inch ;  $2 \times \sqrt{3}$  tons per square inch.

20. A bar of wrought-iron, 5 feet long, has to transmit shocks of 100 foot lbs. without injuring its elasticity. If the limit of elasticity is 30,000 lbs. per square inch, and the modulus of elasticity is 25,000,000 lbs. per square inch, find the sectional area of the bar.

*Ans.* 1·1 square inches.

21. A tie-bar of wrought-iron, 25 feet long, has to resist a shock whose energy is 200 foot lbs. If the modulus of elasticity and working stress are respectively 28,000,000 lbs. per square inch and 10,000 lbs. per square inch, determine the sectional area of the bar.

*Ans.* 4.48 square inches.

22. Through what height must 1 ton fall to produce a stress equal to  $\frac{1}{3}$  of the stress at elastic limit, supposing this latter stress is produced by a shock whose energy is 1 foot-ton?

*Ans.*  $\frac{1}{9}$  of a foot.

23. A cube of 6-inch side is fixed on one face, and a shearing force of 150 tons acts on the opposite face. If the modulus of rigidity equals 5,500 tons per square inch, find the strain, and the work done in distorting the cube.

*Ans.*  $\frac{1}{1320}$ . 64 foot lbs.

24. A bar of mild steel, 1 square inch in sectional area and 10 inches long, is extended by a pull by an amount 0.3 inch. Find the lateral contraction.

$E = 13,600$  tons per square inch.

$C = 5,200$  „ „

$K = 11,900$  „ „

$$\frac{\text{Lateral strain}}{\text{Longitudinal strain}} = \frac{3K - 2C}{6K + 2C}$$

*Ans.* 0.0092.

25. If two tensions of 8 tons per square inch act on a plane, and two compressions of equal magnitude act on a plane perpendicular to the first, find the strain in the direction of each stress. Take the same values of constants as given in Exercise 24.

*Ans.*  $\frac{1}{1298}$ .

26. Stresses of 7 tons per square inch and 5 tons per square inch respectively act normally on two planes at right angles to each other. If both stresses are tensile, find the total strain in the direction of each stress. Use same constants as in Exercise 24.

*Ans.* .0004; .0002.

27. A weight of 20 tons is attached to a bar 4 square inches in sectional area. Find the intensity of stress on a plane making an angle of  $30^\circ$  with the cross-section.

*Ans.* 4.33 tons per square inch.

28. A bar, 4 square inches in sectional area, carries a weight of 20 tons. Find the normal and tangential stresses on a plane whose normal makes an angle of  $60^\circ$  with the axis of stress.

*Ans.* 1.25 tons per square inch.

2.165 „ „

29. Given that the principal stresses at a point are 6 and 5 tons per square inch, both tensile, find the normal stress and tangential stress on a plane whose normal makes an angle  $30^\circ$  with the first stress. Determine also the resultant stress on this plane.

*Ans.* 5.75 tons per square inch.

0.433        „        „

5.76        „        „

30. If in last exercise the stress of 6 tons per square inch is tensile, and the stress of 5 tons per square inch is compressive, find the normal and tangential stresses on the plane.

*Ans.* 3.25 tons per square inch.

4.763        „        „

31. Given that the principal stresses at a point are 6 tons per square inch *tensile*, and 6 tons per square inch *compressive*, find the normal, tangential, and resultant stresses on a plane whose normal is inclined at  $60^\circ$  to the first stress.

*Ans.* 3 tons per square inch.

5.1        „        „

6        „        „

32. In last exercise (31) find the same stresses on a plane whose normal makes an angle of  $45^\circ$  with the first stress.

*Ans.* 0 ; 6 tons per square inch.

33. At a point within a strained solid the stress on one plane is a tension of 80 lbs. per square inch and inclined to the normal at  $25^\circ$  ; the normal component of the stress on a second plane through it at right angles to the first plane is 50 lbs. per square inch tensile. Find the total stress on the second plane in magnitude and direction.

*Ans.* 60.36 lbs. per square inch inclined at  $34^\circ 10'$  to the normal.

34. At a point within a solid in a state of strain the stresses on a pair of rectangular planes through it are—on  $AB$  a normal stress of 250 lbs. per square inch ; on  $CD$  a normal stress of 200 lbs. per square inch. The tangential stresses on each plane are of intensity 31 lbs. per square inch. Find the planes of principal stress.

*Ans.* The inclinations of planes of principal stress to  $AB = 25^\circ 34'$  and  $115^\circ 34'$ .

35. The stresses on a steel bar normal to its cross-section vary from a maximum tension of 30 tons to a minimum tension of 12 tons. Determine the working stress, and the necessary sectional area.

*Ans.* 7.2 tons per square inch ; 4.2 square inches.

36. The stresses on a wrought-iron bar normal to its cross-section vary between a tension of 24 tons and a compression of 12 tons. Find the sectional area.

*Ans.* 7.3 square inches.

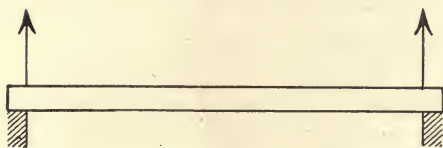
## CHAPTER V.

### BENDING. BENDING MOMENTS AND SHEARING FORCES.

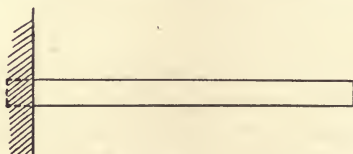
#### 44. Bending.

*Beams.* A beam is the name given to any member of a structure which is exposed to *transverse* stresses. The term *girder* is usually applied to beams made of iron or steel, of a flanged form—that is, consisting of a top and bottom flange connected by a web.

A *beam* or *girder* is usually <sup>either</sup> supported or fixed at the extremities and loaded at points between them.



A *cantilever* is a beam or girder fixed or *encastré* at one end, and free at the other.



A *continuous beam* or girder is one supported at three or more points.



When a beam rests on supports and is loaded with weights acting vertically downwards, the upward or *supporting forces* at the points of support are called *reactions*.

Forces which act upwards are considered positive, and those which act downwards negative.

#### 45. Loaded beams.

The following conditions are assumed :—(1) That the unstrained beam is straight and has a longitudinal plane of symmetry. (2) The bending forces are applied in that plane normally to the axis. (3) That plane parallel normal sections of the unstrained beam remain plane and normal after bending, radiating to the centre of curvature.

When a beam is acted on by any load, the fibres on one side of the beam are stretched, and those on the other compressed. If the beam is supported at the ends, the lower fibres will be stretched and upper fibres compressed. In a cantilever the upper fibres will be stretched and the lower ones compressed. The surface which separates these two portions of the beam, and which is neither extended nor compressed, is called the *neutral surface* of the beam, and the line where the neutral surface intersects any cross-section of the beam is called the *neutral axis* of that cross-section.

In Fig. 78,  $AB$  is the neutral surface.

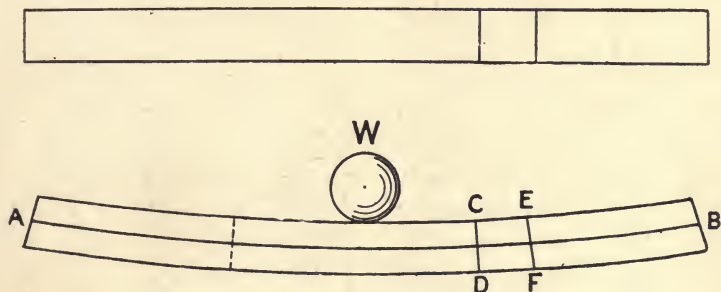


Fig. 78.

#### 46. Stress at each point varies as its distance from the neutral axis.

We shall assume that the strains lie within the elastic limit, and consequently the strains and stresses follow Hooke's law.

In Fig. 80, a longitudinal section of a small portion of beam, let  $CD$  and  $EF$  be two plane cross-sections, taken very close together, which are parallel before straining, and become inclined to one another when the beam is strained, radiating towards the centre of curvature  $O$ .  $NM$  is the portion of neutral surface between the two cross-sections; it remains unaltered in length, and the intersection of this surface with cross-section of beam is called the *neutral axis*, i.e.  $PQ$  of Fig. 79, which shows the cross-section of beam.

Let  $L$  (Fig. 80) be the original distance  $NM$  between the two cross-sections;  $R$  the radius of curvature of the neutral surface.

Draw  $GH$  through  $M$ , parallel to  $CD$ .

Now if we consider any layer  $SJ$ , which is at a distance  $y$  from the neutral surface  $NM$ , or the neutral axis  $PQ$ , this layer has altered in length from  $NM = SK$  to  $SJ$ .

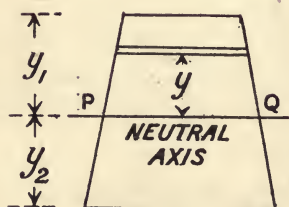


Fig. 79.



Fig. 80.

Calling this alteration of length ( $JK$ )  $l$ , the strain  $= \frac{l}{L}$ .

This longitudinal strain must be accompanied by a longitudinal stress, and if  $f$  is the intensity of that stress,  $f = E \frac{l}{L}$ .

Further  $\frac{l}{y} = \frac{L}{R}$  by similar triangles.

Hence  $f = \frac{E}{R} \cdot y$ .....(1),

where  $E$  is Young's modulus of elasticity, which we suppose the same for all fibres of the beam.

Thus, the stress at any point of the cross-section of the beam is

proportional to its distance from the neutral axis—that is, a uniformly-varying-stress.

#### 47. Stress in beams.

Let Fig. 81 represent a beam supported at both ends, and loaded with weights  $W_1$ ,  $W_2$ ,  $W_3$ . Let  $R_1$  and  $R_2$  be the reactions at the supports. Take  $PQ$  any normal cross-section and consider the separate equilibrium of the portion of the beam on *either* side of the cross-section. Now, stress being the internal resistance to deformation, there must be equilibrium between the internal stresses and the external forces. Thus the portion  $B$  of the beam is held in equilibrium by the external forces  $R_1$  and  $W_1$ , and the stresses which  $A$  exerts on  $B$ . Consider the equilibrium of the portion  $B$ , and for clearness imagine the two portions of the beam  $A$  and  $B$  to be separated at the section  $PQ$ , as shown enlarged in Fig. 82, and replace  $A$  by the stresses which it exerts at the section.

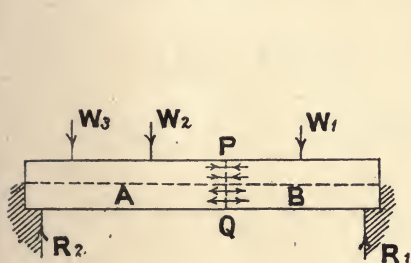


Fig. 81.

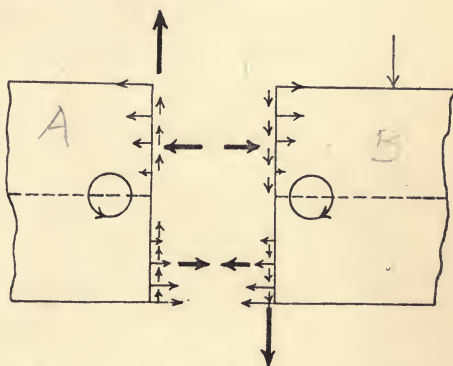


Fig. 82.

The three statical conditions of equilibrium are:

- (1) The sum of the vertical components of stress must be equal to the sum of the vertical components of the external forces.
- (2) The sum of the horizontal components of stress must be equal to the sum of the horizontal components of the external forces.
- (3) The sum of the moments of stress about any axis must be equal to the sum of the moments of the external forces about the same axis.

**SHEARING FORCE.** By condition (1), since the portion  $B$  is in equilibrium, and the loads acting on it are all *vertical*, the sum of the vertical components of stress in the downward direction is

$$= R_1 - W_1,$$

which is called the *shearing force*. It tends to shear  $B$  from  $A$ .

Similarly, by considering the equilibrium of the portion  $A$  we get the shearing force

$$= R_2 - W_2 - W_3.$$

These shearing forces, the one on the right-hand portion of the beam, the other on the left-hand portion, are *equal in magnitude but opposite in sign*.

Next, consider the horizontal equilibrium of the portion  $B$ . The *external forces* being all vertical have no horizontal component. As to the horizontal stresses above the neutral axis the portion  $A$  tends to push  $B$  to the right, but below the neutral axis  $A$  tends to pull  $B$  to the left. For equilibrium it is necessary that the *total push should be equal to the total pull*, or the total horizontal force at the section be zero. Thus the horizontal components of stress constitute a couple.

#### 48. Position of the neutral axis.

Now if we consider a very small strip of the cross-section of area " $a$ ," distant  $y$  from the neutral axis, Fig. 83, the total push or pull on this small area is

$$= fa,$$

where  $f$  is the intensity of stress on the area " $a$ ."

Summing up the forces on all these elements of area composing the cross-section, we must have according to the above condition of equilibrium

$$\Sigma fa = 0,$$

or, since

$$f = \frac{E y}{R},$$

$$\frac{E}{R} \Sigma ya = 0;$$

or,

$$\Sigma ya = 0,$$

as  $\frac{E}{R}$  is constant for all such elements of area.

Thus, the sum of the products of each element of area into its distance from the neutral axis is zero; and this can be true only if the *neutral axis passes through the centre of gravity of the section*.

#### 49. Bending moment and Moment of resistance.

The third condition of equilibrium is that the sum of the moments of all the internal stresses about any axis must balance the moments of the external forces about the same axis.

Again, considering the equilibrium of either portion of the beam,  $A$  or  $B$ , on one side of the section  $PQ$ , Fig. 81, the *bending moment* is

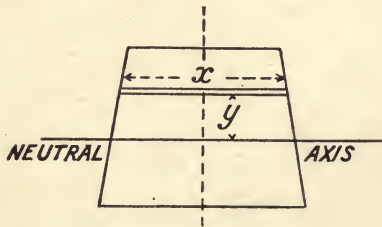


Fig. 83.

the algebraic sum of the moments about an axis in the section, of the external forces on one side of the section. The moment of the resisting stress couple may be taken about any axis; for convenience it is taken about the neutral axis, and this moment is called the *Moment of Resistance*, or, the *Moment of the Elastic Forces*, and is equal in magnitude, but opposite in sign, to the corresponding bending moment due to the external forces.

Let  $l_1, l_2$ , be the perpendicular distances from the section  $PQ$  to  $R_1$  and  $R_2$  respectively; and let  $x_1, x_2, x_3$ , be the distances from  $PQ$  to  $W_1, W_2, W_3$  respectively.

The bending moment with respect to  $PQ$  of all the forces on the right of  $PQ$  is

$$R_1 l_1 - W_1 x_1,$$

and this is equal to the moment of the stress couple exerted on the portion  $B$  at the section  $PQ$ .

The bending moment with respect to  $PQ$  of all the forces on the left of  $PQ$  is

$$R_2 l_2 - W_2 x_2 - W_3 x_3;$$

this is equal to the moment of the stress couple exerted on the portion  $A$  at the section  $PQ$ ; and hence is equal in magnitude to the moment

$$R_1 l_1 - W_1 x_1.$$

It should be noted that the shearing force and bending moment *change sign*, according as we consider the equilibrium of one portion or the other of the beam at the section  $PQ$ .

In order to determine a relation between the bending moment and the stress produced by it, consider as before the elementary strip of area  $a$  distant  $y$  from the neutral axis on which the intensity of stress is  $f$ .

The force on this element of area is  $fa$ .

Its moment about neutral axis is  $fa y = \frac{E}{R} ay^2$ .

The *total moment of the stresses* for whole area of section

$$= \frac{E}{R} \Sigma ay^2.$$

The term  $\Sigma ay^2$ , or the sum of all the small elements of area each multiplied by the square of its distance from the neutral axis, is termed the *Moment of Inertia* of the cross-section about that axis, and is usually denoted by the symbol  $I$ . Let  $M$  = the bending moment at the section.

For equilibrium we must therefore have

$$M = \frac{E}{R} \Sigma ay^2 = \frac{E}{R} I \dots\dots\dots(2),$$

or, since  $\frac{E}{R} = \frac{f}{y}$  by equation (1),

$$M = \frac{f}{y} I \dots\dots\dots(3).$$

Thus, the bending moment produces a uniformly varying longitudinal stress the intensity of which  $f$  at distance  $y$  from the neutral axis is

$$f = \frac{My}{I}.$$

If  $y_1, y_2$  be the extreme distances from the neutral axis to the top and bottom edges of the cross-section, and  $f_1, f_2$  the maximum stresses corresponding to  $y_1$  and  $y_2$ ,

$$M = \frac{f_1}{y_1} I = \frac{f_2}{y_2} I.$$

The strength of the beam, or maximum moment of resistance to bending, is determined from this equation.

$R$  is the *Radius of Curvature* of the bent beam at the place under consideration, and the *curvature* can be expressed as

$$\frac{1}{R} = \frac{f}{Ey} \quad \text{or} \quad \frac{1}{R} = \frac{M}{EI}.$$

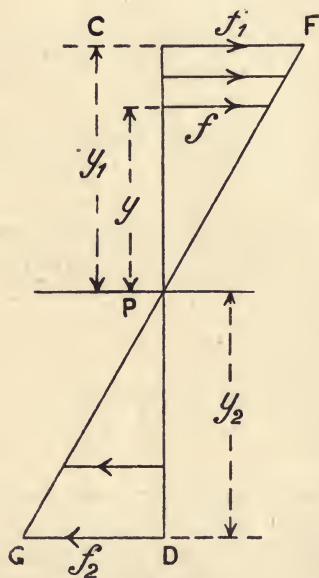


Fig. 84.

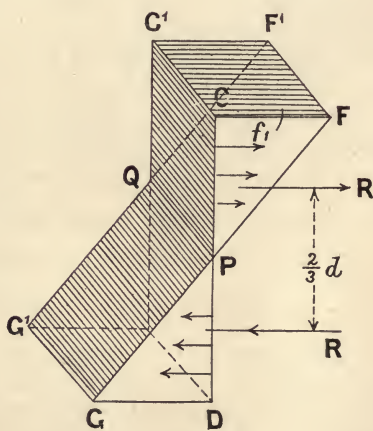


Fig. 85.

The stress due to bending may be graphically represented as in Fig. 84, where  $CD$  is the side elevation of the cross-section on which the stress acts. The neutral axis is at right angles to  $CD$  through  $P$ , the centre of gravity of the cross-section. The stress varies uniformly from  $P$  in direction  $CD$ . The greatest intensity of compressive stress  $f_1 (= CF)$  occurs at  $C$ , and the greatest intensity of tensile stress

$f_2 (=GD)$  occurs at  $D$ , and the stress at any point of the section is equal to the horizontal ordinate to the inclined line  $FG$ . Hence the stresses may, as regards the whole cross-section of the beam, be represented by a wedge-shaped figure (Fig. 85), where  $PQ$  represents the width of the beam and  $CD$  the depth. The resultant stresses above and below the neutral axis will be represented by the volume of the wedge-shaped figures above and below that axis respectively, and these resultants act at right angles to  $CD$  through the centres of gravity of the triangles. To find the moment of resistance we can either consider these forces with a leverage equal to the distance between the two, or consider the effect of both round the neutral axis. In either case we get for a rectangular cross-section

$$\text{Moment of Resistance} = \frac{f_1}{2} \times \frac{d}{2} \times b \times \frac{2}{3} d = \frac{1}{6} f_1 b d^2,$$

where  $f_1$  is the intensity of stress at  $C (=CF)$ ,  $PQ = b$ , and  $CP = \frac{d}{2}$ .

GRAPHICAL REPRESENTATION IN THE CASE OF RECTANGULAR BEAMS. It is evident that if we take the cross-section of a rectangular beam and draw the diagonals  $CD'$  and  $C'D$ , we have a *graphic representation* of the stresses from the neutral axis to the edges, and the shaded areas  $COC'$  and  $DOD'$  represent the total tension and compression. For if in Fig. 86,  $CC'$  represents on any scale the intensity of stress at the top edge, distant  $y_1$  inches from the neutral axis, then stress intensity  $y$  inches from neutral axis is

$$CC' \times \frac{y}{y_1} = ab.$$

Hence the triangles  $COC'$  and  $DOD'$  represent areas of equal resistance above and below the neutral axis, and graphically represent the quantity and distribution of the resistance.

The moment of resistance can be obtained from these triangles as before. If  $CC' = b$ ,  $CD = d$ , the area of the triangle  $COC' = \frac{bd}{4}$ , and the leverage of the  $M_R = \frac{2}{3}d$ .

Hence 
$$M_R = f_1 \times \frac{bd}{4} \times \frac{2}{3} d = \frac{1}{6} f_1 b d^2.$$

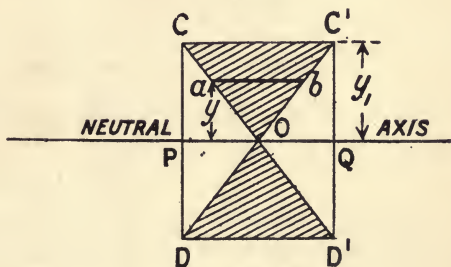


Fig. 86.

**BEAMS OF RECTANGULAR AND I SECTION.** As the extreme fibres in rectangular beams are the only parts of the beam exerting their full working resistance, the rest of the beam might be proportionately reduced in section. In timber beams this reduction of section is not carried out in practice; but in iron or steel beams this principle is carried into effect, and beams of these metals are rolled in the shape of letter I, or else built up of plates and angle-irons, concentrating the material at the top and bottom edges of the cross-section in horizontal *flanges*, which latter are connected by a vertical part or *web*.

### 50. Case of Simple Bending.

In this simple case, as illustrated in Fig. 87, the beam is acted on by two *equal* and *opposite* couples. The reaction at each support is

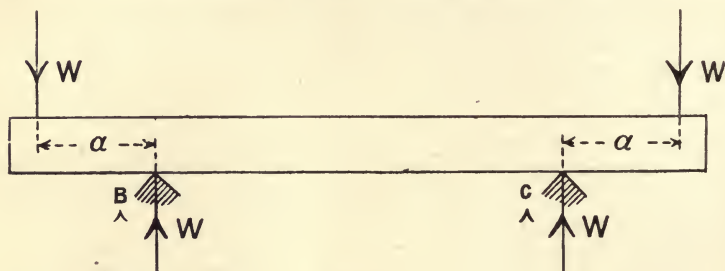


Fig. 87.

equal to  $W$ , and there is *no shearing force* between  $B$  and  $C$ ; for on any section  $PQ$  this force equals

$$W - W = 0.$$

Again, between  $B$  and  $C$  the *bending moment* is *constant*, being everywhere equal to  $Wa$ ; and the curvature is therefore *circular*.

The stress at any section is a couple whose moment is  $Wa$ .

### 51. Bending moments and shearing forces on cantilevers and beams.

In the case of a beam supported at both ends and loaded the fibres above the neutral axis are in compression and are shortened, while those below are in tension and are lengthened; so that the centre of curvature is *above* the beam. In the cantilever the reverse takes place.

We shall consider the bending moments in the first case positive, while in the cantilever they are negative. In continuous beams, as will be seen later on, the bending moments are partly positive and partly negative.

A shearing force at a section is considered positive when it tends to shear the right-hand portion of the beam upwards.

When a *beam* is acted on by external loads, the *bending moment* at a given section is equal to the algebraic sum of the moments about the section of all the external forces acting on the portion of the beam on either side of it; and the *shearing force* at any section is the algebraic sum of all the external forces acting on either portion of the beam into which the section divides it. We shall usually consider the right-hand portion of the beam.

In a *cantilever* the *bending moment* at a given section is equal to the sum of the moments about the section of all the loads between the section and the free end of the cantilever; and the shearing force at any section is equal to the sum of the loads between the section and the free end.

In representing bending moments and shearing forces graphically, when positive they are measured upwards above a horizontal datum line, and when negative they are measured downwards.

CASE 1. CANTILEVER LOADED WITH A SINGLE WEIGHT AT THE FREE END.

Let Fig. 88 represent a cantilever of length  $l$  fixed tangentially at  $A$  and loaded with a weight  $W$  at the free end  $B$ .

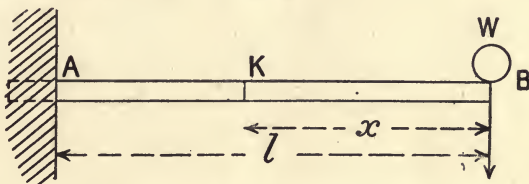


Fig. 88.

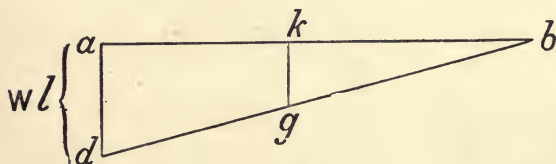


Fig. 89.

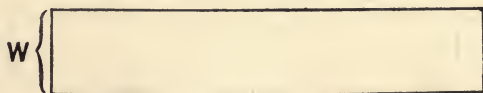


Fig. 90.

At any section distant  $x$  from the free end the bending moment

$$M_x = -Wx$$

and the shearing force

$$F_x = -W.$$

The bending moment evidently varies as  $x$ , and at the fixed end, where  $x=l$ , its value is

$$\text{Max. } M = -Wl.$$

The diagram of bending moment is a triangle as shown in Fig. 89, where  $ad = Wl$ . The moment at any section  $K$  is represented by the ordinate  $kg$ .

The diagram of shearing force is a rectangle as in Fig. 90, all ordinates being equal to  $W$ .

It is well to note that the ordinates in the bending-moment diagram represent expressions such as inch-tons or foot-lbs., whereas the ordinates in the shearing-force diagram represent simply lbs. or tons.

#### CASE 2. CANTILEVER WITH SEVERAL LOADS.

Let  $AB$ , Fig. 91, be the cantilever fixed at  $A$ , and loaded with weights  $W_1$ ,  $W_2$ ,  $W_3$ , applied at distances  $x_1$ ,  $x_2$ ,  $x_3$  from the fixed end.

The bending moment at any section  $K$ , distant  $x$  from the fixed end, is equal to

$$M_x = -W_1(x_1 - x) - W_2(x_2 - x).$$

Fig. 91.

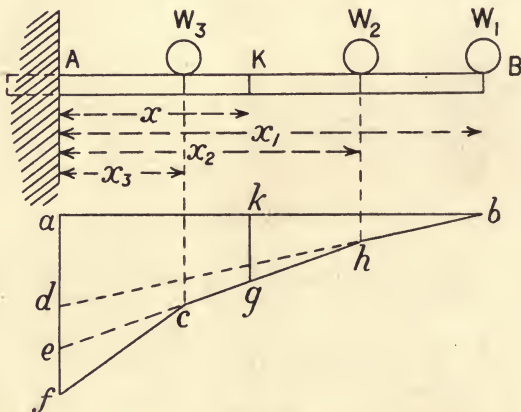


Fig. 92.

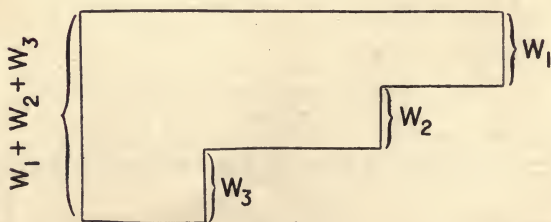


Fig. 93.

The maximum bending moment occurs at the fixed end and is equal to

$$M_A = -(W_1x_1 + W_2x_2 + W_3x_3).$$

The shearing force at section  $K$

$$F = -(W_1 + W_2).$$

The shearing force at the fixed end

$$= -(W_1 + W_2 + W_3).$$

To draw the diagram of bending moments, Fig. 92, let  $ab$  represent the cantilever; draw  $ad$  at right angles to  $ab$  to represent on any scale of moments the moment of  $W_1$  about  $A$ , *i.e.*  $W_1x_1$ ; produce  $ad$  to  $f$ , making  $de$  equal to  $W_2x_2$ , and, similarly,  $ef$  equal to  $W_3x_3$ , on the same scale of moments. Let  $h$  be the point on  $bd$  vertically below  $W_2$ , and  $c$  the point on  $eh$  vertically below  $W_3$ ; the polygon  $abhcf$  will represent the diagram of moments for the whole cantilever, and the bending moment at any section  $K$  will be found by measuring the ordinate  $kg$  of the polygon.

The shearing-force diagram is shown in Fig. 93.

The vertical ordinate at any section to the stepped figure gives the value of the shearing force at that section.

CASE 3. CANTILEVER LOADED UNIFORMLY OVER ITS WHOLE LENGTH.

Let  $l$  be the length of the cantilever, and  $w$  the uniformly-distributed load per unit of length. The total load  $W = wl$ .

In the case of a uniform load its weight may be assumed as acting at its centre of gravity.

The bending moment at a section  $K$  distant  $x$  from the free end is

$$M_x = -wx \times \frac{x}{2} = -\frac{wx^2}{2}.$$

The maximum bending moment occurs at the fixed end, where  $x = l$  and  $M_A = -\frac{wl^2}{2}$ .

As  $M_x$  varies as the square of the distance, the successive moments may be represented by the *ordinates of a parabola*, of which the free end of the cantilever is the vertex.

The shearing force at section  $K$  is equal to

$$F_x = -wx.$$

Its maximum value at the fixed end,

$$F_A = -wl = -W.$$

To draw the diagram of bending moments, Fig. 94, make  $ad$  on any scale of moments equal to  $\frac{wl^2}{2} = \frac{Wl}{2}$ . From  $d$  draw a parabola passing through  $d$  and touching  $ab$  at  $b$ . The bending moment at any section  $K$  is found by measuring the ordinate  $kg$  to the parabola  $bd$ .

The diagram of shearing force is a triangle as shown in Fig. 95.

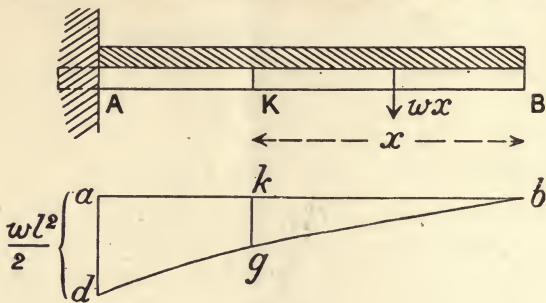


Fig. 94.



Fig. 95.

CASE 4. CANTILEVER LOADED UNIFORMLY OVER PART OF ITS LENGTH.

Let  $a$  be the length of the cantilever loaded with a uniform load  $w$  per unit of length. Then,  $x$  being measured from the free end

For  $x > a$ ,  $M_x = -wa \left( x - \frac{a}{2} \right)$ .

Maximum at  $x = l$ ,  $M_A = -wa \left( l - \frac{a}{2} \right)$ .

For  $x < a$ ,  $M_x = -\frac{wx^2}{2}$ .

Shearing force

For  $x > a$ ,  $x = a$ ,  $F_x = -wa$ .

For  $x < a$ ,  $F_x = -wx$ .

To draw the diagram of bending moments, Fig. 96, consider the

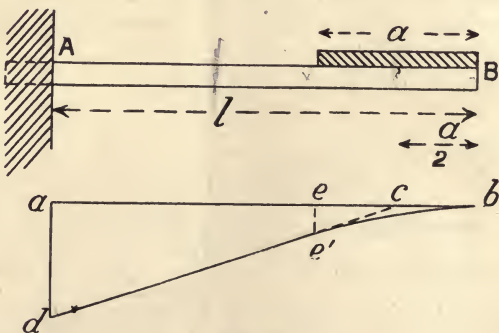


Fig. 96.

load as concentrated at its centre of gravity and lay off  $ad$  as for a single load  $wa$  at  $c$ , the ordinate  $ee'$  being equal to  $\frac{wa^2}{2}$ . Draw the straight line  $ce'd$  and the parabola  $be'$ ; then  $be'd$  will be the curve of moments. The shearing-force diagram is shown in Fig. 97.



Fig. 97.

CASE 5. CANTILEVER LOADED WITH A WEIGHT AT THE FREE END, AND WITH A UNIFORM LOAD.

Let  $W$  be the weight at the free end, and  $w$  the uniform load per unit of length. Then

$$M_x = - \left( Wx + \frac{wx^2}{2} \right),$$

and maximum bending moment

$$M_A = - \left( Wl + \frac{wl^2}{2} \right).$$

The shearing force at any section,

$$F_x = - (W + wx).$$

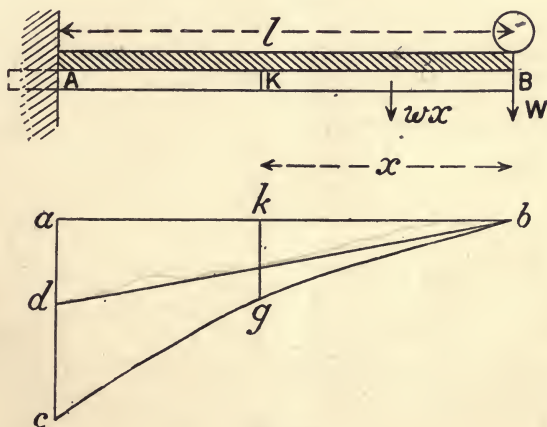


Fig. 98.

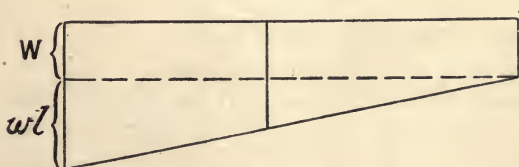


Fig. 99.

The maximum value at fixed end,

$$F_A = -(W + wl).$$

*Diagrams.* The curves of bending moment and shearing force are shown in Figs. 98 and 99, the ordinates of which in each case are equal to the *sum* of the ordinates of the diagrams for each load taken separately.

CASE 6. CANTILEVERS PROJECTING OVER ONE SUPPORT, LOADED WITH A WEIGHT  $W$  AT ONE END, AND ANCHORED DOWN AT THE OTHER. (Fig. 100.)

Let  $P$  be the force acting at  $C$ ,  $R$  the reaction at  $A$ . Then

$$P \times CA = W \times AB.$$

In Fig. 101 if  $ad$  is set down to represent the moment  $W \times AB$ , then  $cdb$  is the diagram of bending moments; or, for illustration, we may consider the moment  $W \times BC$ . Set down  $ce$  on the same scale as before to represent  $W \times BC$ ; this must also be equal to  $R \times AC$  and the triangle  $ced$  will be the diagram of moments for the reaction  $R$ ; the ordinates  $fg$ , &c., being measured *upwards* from  $ed$ , and the bending moment at any section  $K$  (being the difference of the moments of  $W$  and  $R$ ) will be  $kg$ , the difference between the ordinates  $kf$  and  $fg$ .

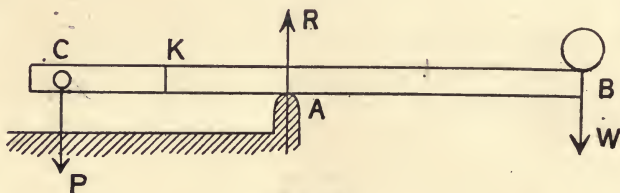


Fig. 100.

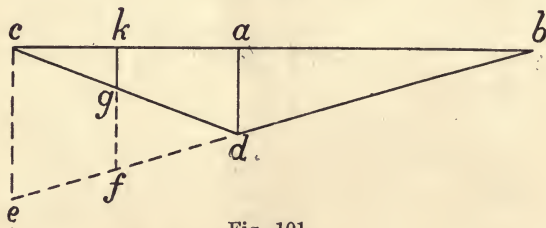


Fig. 101.

## 52. Beams supported at both ends.

To find the bending moment at any section:—Take the algebraic sum of the moments (about the section) due to the reaction of either support and to any loads between that support and the section in question.

The shearing force at any section is equal to the algebraic sum of

all the external forces (including the reaction) acting on either side of the section.

The first thing to be done is to find the reactions. In symmetrically-loaded beams each reaction is evidently equal to half the total weight; in other cases the reaction at either support is obtained by taking moments round the opposite support. The sum of the reactions must be equal to the total load on the beam.

CASE 7. BEAMS SUPPORTED AT BOTH ENDS AND LOADED WITH A SINGLE WEIGHT.

Let  $AB$  (Fig. 102) be the beam loaded with a weight  $W$  at  $C$ .

Let  $AB = l$ ;  $AC = a$ .

$$\text{Reaction at } A = R_2 = W \frac{l-a}{l}.$$

$$\text{Reaction at } B = R_1 = W \frac{a}{l}.$$

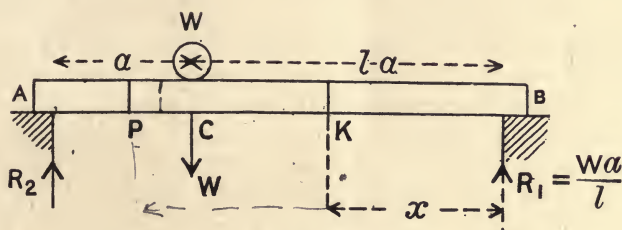


Fig. 102.

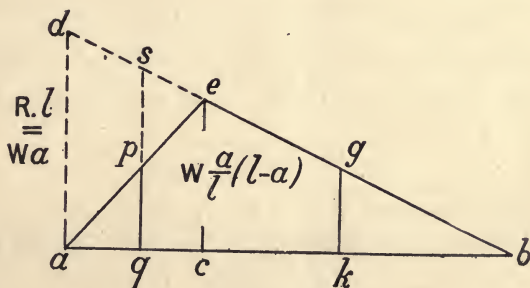


Fig. 103.

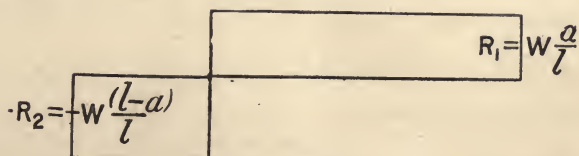


Fig. 104.

The bending moment at any section  $K$  between  $B$  and  $C$  is

$$M_x = R_1 x; \quad = W \frac{a}{l} x.$$

The maximum bending moment occurs at  $C$ , where the load acts, and here

$$M = R_1 \times (l - a) = W \frac{a}{l} (l - a).$$

For any section between  $A$  and  $C$  the bending moment is

$$R_1 x - W[x - (l - a)] = W \frac{a}{l} x - Wx + W(l - a),$$

or it is  $R_2(l - x)$ , which

$$= \frac{W(l - a)}{l} (l - x) = W \frac{a}{l} x - Wx + W(l - a).$$

Fig. 103 gives the graphical representation, where  $aeb$  is the diagram of moments,  $ce$  representing on any scale the moment  $\frac{Wa(l - a)}{l}$ , and the moment at any section  $K$  is the ordinate  $kg$ .

This diagram can also be got by considering the contrary moments of  $R_1$  and  $W$  about  $A$ . In figure  $ad$  represents the moment of  $R_1$  round  $A$ , and also the moment of  $W$  round  $A$ , these moments being equal. The bending moment at any point  $P$  is  $sq - sp = pq$ .

The shearing-force diagram is shown in Fig. 104, where the positive shearing-force in the right-hand portion is equal to  $R_1$ , and the negative shearing-force in the left-hand portion is equal to  $R_2$ , i.e.  $R_1 - W$ .

CASE 8. SINGLE LOAD  $W$  AT THE CENTRE OF THE SPAN.

The reactions are each  $= \frac{W}{2}$ , and the bending moment at any section distant  $x$  from  $B$  is  $M = R_1 x$ , and the shearing force

$$F = R_1 = \frac{W}{2}.$$

For any section between  $W$  and  $R_2$ ,  $M = R_2(l - x)$ .

The maximum bending moment is at the centre, and its value is

$$R_1 \frac{l}{2} = \frac{Wl}{4}.$$

The diagrams of bending moment and shearing force are sketched in Figs. 105 and 106.

In Fig. 105  $ad$  represents the moment of  $R_1$  round  $A = \frac{Wl}{2}$ ; then, completing the triangle, we see that

$$cc' = \frac{ad}{2} = \frac{Wl}{4}.$$

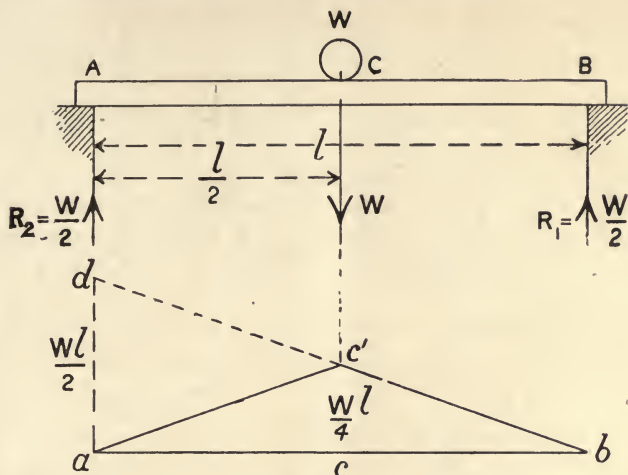


Fig. 105.



Fig. 106.

CASE 9. BEAM SUPPORTED AT BOTH ENDS AND LOADED WITH TWO OR MORE SEPARATE LOADS.

Let  $l$  be the length of the span, and  $W_1, W_2, W_3$  the loads at distances  $a_1, a_2, a_3$  respectively from the left support (Fig. 107).

$$\text{Then } R_1 = \frac{W_1 a_1 + W_2 a_2 + W_3 a_3}{l};$$

$$R_2 = \frac{W_1 (l - a_1) + W_2 (l - a_2) + W_3 (l - a_3)}{l}.$$

To find the bending moment at any section  $K$  distant  $x$  from the right support, let  $x_1, x_2, x_3$  represent the distances of the several weights from this section. Then, considering the right-hand portion, we get

$$M = R_1 x - W_3 x_3 - W_2 x_2.$$

The diagram of bending moments may be drawn by drawing the diagrams for each load separately, in the first instance, and then combining them by adding the ordinates as in Fig. 107, where  $cc' = ch + cg + cf$ ; or, second method, by calculating the bending moment at the point of application of each weight when all the weights rest on the beam, and drawing the ordinates  $cc', dd', ee'$

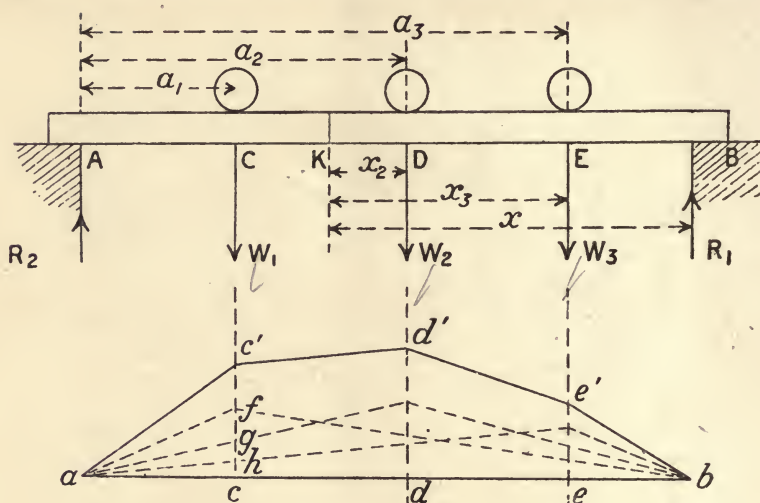


Fig. 107.

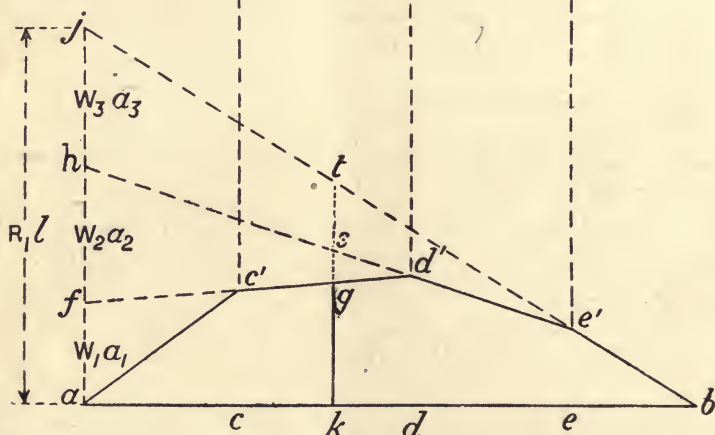


Fig. 108.

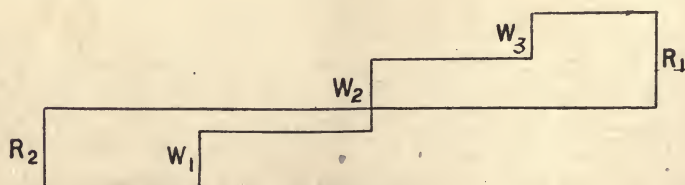


Fig. 109.

(Fig. 107), equal to the moments at  $C$ ,  $D$ , and  $E$ ; then  $ac'd'e'b$  will be the diagram for beam; or, the following purely graphic method (Fig. 108).

From  $a$  draw the ordinate  $aj = R_1 l$ ,  $l$  being the span, and join  $jb$ , on the line  $aj$  mark off  $af = W_1 a_1$ ,  $fh = W_2 a_2$ ,  $hj = W_3 a_3$ . The sum of these three ordinates must be equal to  $R_1 l$ , since  $\Sigma Wa = R_1 l$ , as there is no bending moment at the pier  $A$ . Join  $h$  with  $e'$ , the point where  $jb$  meets the line of  $W_3$ ;  $f$  with  $d'$  the point where  $he'$  meets the line of  $W_2$ ; let the line of  $W_1$  meet  $fd'$  in  $c'$ , then  $ac'd'e'b$  is the diagram of moments.

The bending moment at any section  $K$  is given by the ordinate  $kg$ , for at this section

$$M = R_1 x - W_3 x_3 - W_2 x_2 = kt - ts - sg = kg.$$

The shearing force diagram is shown in Fig. 109.

CASE 10. BEAM SUPPORTED AT BOTH ENDS, AND LOADED THROUGHOUT ITS LENGTH WITH A UNIFORMLY-DISTRIBUTED LOAD.

Let  $l$  be the length of the beam,  $w$  the uniform load per foot run then, if  $W$  is the total weight,  $W = wl$ . The reaction at each support is  $\frac{wl}{2}$ . At any section distant  $x$  from the right-hand end the bending moment

$$M = R_1 x - wx \cdot \frac{x}{2} = \frac{w}{2} \cdot x(l - x) \dots\dots\dots(1).$$

This is a maximum when  $x = \frac{l}{2}$ ,

$$\left(\frac{dM}{dx} = 0 \text{ when } x = \frac{l}{2}\right);$$

$$\therefore \text{Max. } M = \frac{wl^2}{8} = \frac{Wl}{8}.$$

Equation (1) is that of a parabola, the origin of coordinates being at a point on the curve distant  $\frac{l}{2}$  from the axis; hence to draw the diagram of moments (Fig. 110), set up the ordinate  $cd$  from the centre to represent  $\frac{wl^2}{8}$ , and draw a parabola passing through  $adb$ .

The bending moment at any section  $K$  is given by the ordinate  $kg$ .

The bending moment curve can also be drawn by setting up the ordinate  $ae = \frac{wl^2}{2}$ , that is the moment of  $R_1$  about  $A$ . Join  $be$ , then for any value of  $x$  set down from this line  $be$  the ordinate  $fg = \frac{wx^2}{2}$ , then  $g$  is a point on the bending moment curve, and any number of points can be found by taking different values of  $x$ .

The bending moment at  $K$  distant  $x$  from right-hand end is

$$R_1x - \frac{wx^2}{2} = fk - fg = kg.$$

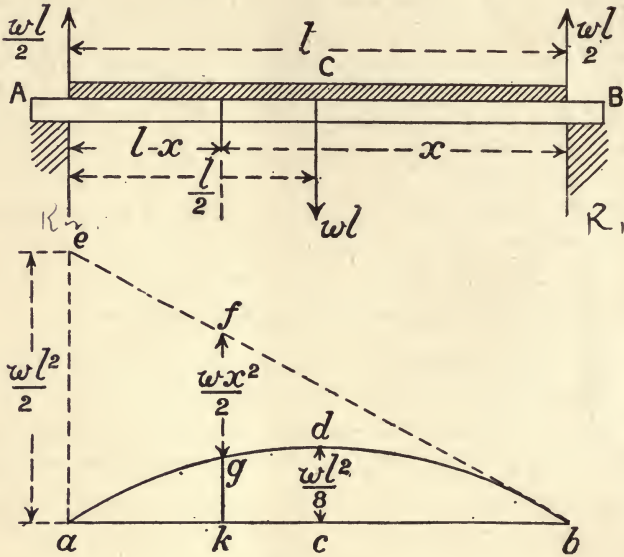


Fig. 110.

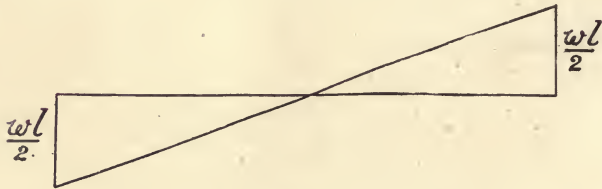


Fig. 111.

The *shearing force* at any section distant  $x$  from the right-hand end is

$$F = R_1 - wx = w \left( \frac{l}{2} - x \right) \dots\dots\dots (2).$$

$F$  is positive when  $x$  is less than  $\frac{l}{2}$ , and negative when  $x$  is greater than  $\frac{l}{2}$ .

At

$$x = 0; \quad F = \frac{wl}{2};$$

$$x = l; \quad F = -\frac{wl}{2};$$

$$x = \frac{l}{2}; \quad F = 0.$$

The diagram of shearing force is sketched in Fig. 111. It may be noted here that the bending moment is a maximum where the shearing force changes sign, since  $\frac{dM}{dx} = F$ , and the value of  $x$  when  $M$  is a maximum is got by equating  $\frac{dM}{dx}$  to zero.

For, as shown,  $M = \frac{w}{2} x (l - x)$ , equation (1);

$$\therefore \frac{dM}{dx} = w \left( \frac{l}{2} - x \right) = F, \text{ equation (2).}$$

CASE 11. BEAM SUPPORTED AT BOTH ENDS AND LOADED WITH A UNIFORM LOAD OVER A PORTION OF ITS LENGTH NEXT TO ONE SUPPORT.

Let  $AB$  be the beam of span  $l$  feet supporting a load of  $w$  per foot distributed over a length  $a$  next the support  $A$ .

Then  $R_1 = \frac{wa^2}{2l}$ ;

$$R_2 = \frac{wa \left( l - \frac{a}{2} \right)}{l}.$$

At any section distant  $x$  from the right support, which includes a portion  $y$  of the uniform load, the bending moment is

$$M = R_1 x - \frac{wy^2}{2}.$$

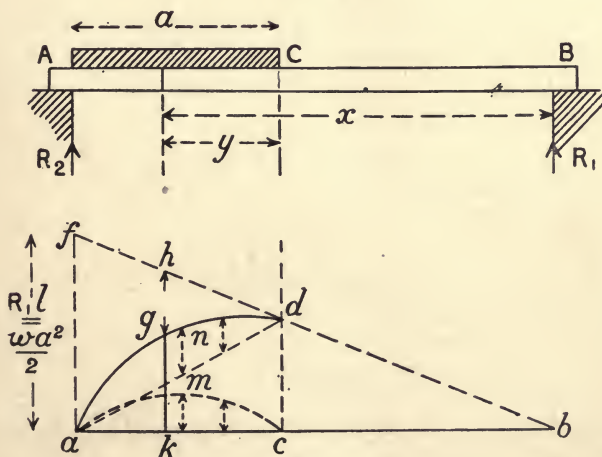


Fig. 112.

To draw the bending moment diagram (Fig. 112), set up the ordinate  $af$  to represent  $R_1 l$ , which also represents  $\frac{wa^2}{2}$ ; join  $fb$ , and

let the vertical through  $C$  meet this line at  $d$ . From the line  $df$  set down the ordinate  $hg$  to represent  $\frac{wy^2}{2}$ , then  $g$  is a point on the bending moment diagram, and a series of points can be thus got for different values of  $y$ .  $agdb$  is the bending moment diagram, of which the portion  $agd$  is a parabola and  $db$  is a straight line.

$$M = R_1x - \frac{wy^2}{2} = kh - hg = kg.$$

Or, another method is to consider the length  $AC$  as if it were a beam supported at  $A$  and  $C$ , and loaded uniformly. On  $ac$  draw the parabola  $amc$  for the uniform load, make  $cd$  equal to the moment of  $R_1$  about  $C$ . Join  $ad$ , and draw the vertical ordinates above  $ad$  equal to the ordinates of  $amc$ .  $agndb$  is the bending moment diagram.

The diagram of shearing force is sketched in Fig. 113.

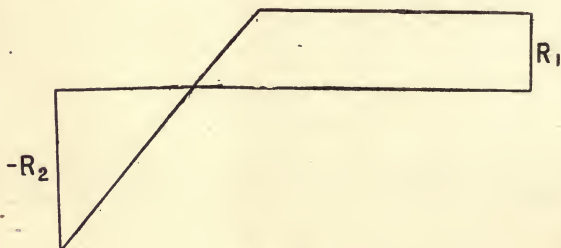


Fig. 113.

CASE 12. BEAM SUPPORTED AT BOTH ENDS AND LOADED UNIFORMLY OVER A PORTION OF ITS LENGTH, NOT EXTENDING TO EITHER ABUTMENT.

Let  $AB$  be the beam of span  $l$ , loaded over the portion  $CD$  with a weight  $w$  per unit of length.

Let  $CD = a$ ,  $BD = z$ .

The total load on beam  $= wa$ .

The reactions are

$$R_1 = wa \frac{l - \left(z + \frac{a}{2}\right)}{l};$$

$$R_2 = wa \frac{z + \frac{a}{2}}{l}.$$

The bending moment at any section between  $B$  and  $D$ , where  $x$  is the distance of the section from right support, is

$$M = R_1x.$$

For any point between  $C$  and  $D$

$$M = R_1x - w \frac{(x - z)^2}{2}.$$

For any point between  $C$  and  $A$

$$M = R_2(l - x) = R_1x - wa \left\{ x - \left( z + \frac{a}{2} \right) \right\}.$$

Diagram of bending moments (Fig. 114).

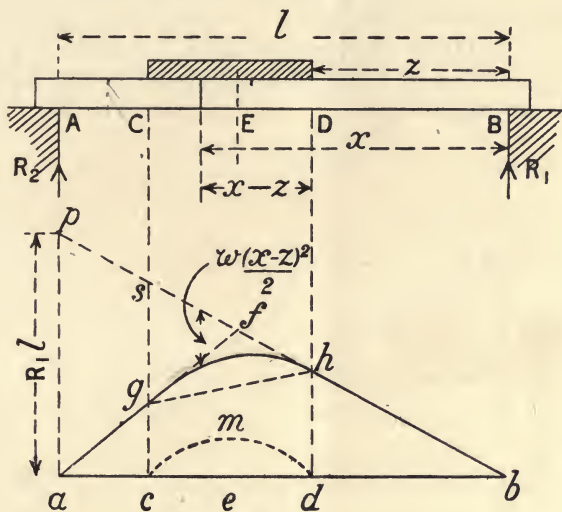


Fig. 114.

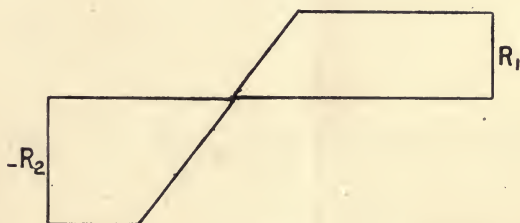


Fig. 115.

Let  $ab$  represent the beam, the load extending over the portion  $cd$ ;  $e$  is the centre of  $cd$ . First consider the weight of the whole load as if it were concentrated at  $E$ , the central point of  $CD$ . Draw the diagram  $afb$  for this load as in Case 7. Then consider the length  $CD$  as if it were a beam supported at  $C$  and  $D$ , loaded uniformly. Draw the parabola  $cmd$  for this load as in Case 10. From  $c$  and  $d$  draw verticals  $cg$  and  $dh$  meeting  $af$  and  $bf$  at  $g$  and  $h$  respectively. Join  $gh$  and draw vertical ordinates above  $gh$  equal to the ordinates of  $cmd$ . Then  $agnhb$  is the diagram of bending moments.

The diagram can also be got as in the previous case, by setting up the ordinate  $ap$  to represent  $R_1l$ . Join  $pb$ , then curved portion of

diagram can be got by setting down from  $h$ s ordinates at different points to represent  $w \frac{(x-z)^2}{2}$ .

The shearing force diagram is shown in Fig. 115.

CASE 13. BEAM LOADED WITH A UNIFORMLY DISTRIBUTED LOAD AND A SINGLE CONCENTRATED LOAD.

Let  $AB$  (Fig. 116) be the beam of length  $l$ , supported at  $A$  and  $B$ , and carrying a uniformly distributed load of intensity  $w$ , together with a weight  $W$  at the point  $C$ .

Let

$$BC = a; \quad CA = b.$$

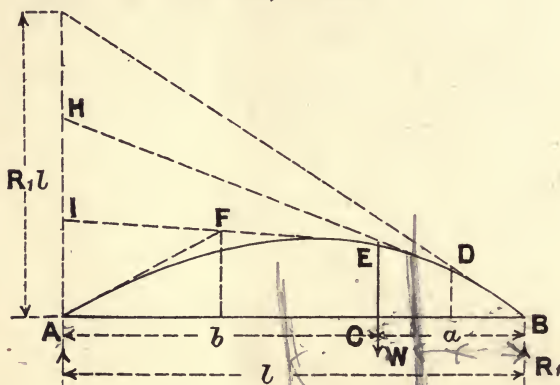


Fig. 116.

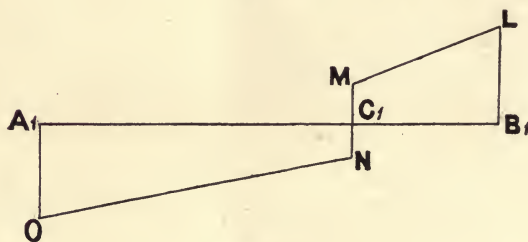


Fig. 117.

In all cases such as this, where the loading is not continuous, but changes abruptly at one or more points of the span, it is necessary to consider separately each portion of the beam between the points of discontinuity.

The reaction at  $B$

$$R_2 = \frac{wl}{2} + W \frac{b}{l}.$$

The shearing force  $F$  at any section between  $B$  and  $C$  distant  $x$  from  $B$  is

$$R_2 - wx = \frac{wl}{2} + W \frac{b}{l} - wx.$$

$$\text{At } B, \quad F = R_1 = \frac{wl}{2} + W \frac{b}{l}.$$

$$\text{At } C, \quad F = \frac{wl}{2} + W \frac{b}{l} - wa.$$

The shearing force,  $F_1$ , at any section between  $C$  and  $A$  distant  $x$  from  $B$  is

$$\begin{aligned} R_1 - W - wx &= \frac{wl}{2} + W \frac{b}{l} - W - wx \\ &= \frac{wl}{2} - W \frac{a}{l} - wx. \end{aligned}$$

$$\text{At } C, \quad F_1 = \frac{wl}{2} - W \frac{a}{l} - wa.$$

$$\text{At } A, \quad F_1 = -\frac{wl}{2} - W \frac{a}{l}.$$

$F_1$  at  $C$  is positive or negative according as  $\frac{wl}{2} \gtrless W \frac{a}{l} + wa$ .

To draw the shearing force diagram (Fig. 117). On the verticals through  $B_1$ ,  $C_1$ ,  $A_1$  set off

$$B_1L \text{ to represent } R_1 = \frac{wl}{2} + W \frac{b}{l}.$$

$$C_1M \text{ „ „ } \frac{wl}{2} + W \frac{b}{l} - wa.$$

$$C_1N \text{ „ „ } \frac{wl}{2} - W \frac{a}{l} - wa.$$

$$A_1O \text{ „ „ } -\frac{wl}{2} - W \frac{a}{l}.$$

Join  $LM$  and  $NO$ ; this completes the diagram.

*Bending moments.* The bending moment at any section between  $B$  and  $C$  is

$$M = R_1x - \frac{wx^2}{2} = \left( \frac{wl}{2} + W \frac{b}{a} \right) x - \frac{wx^2}{2}.$$

$$\text{At } C, \quad M = \left( \frac{wl}{2} + W \frac{b}{a} \right) x - \frac{wa^2}{2}.$$

At any section between  $C$  and  $A$

$$M = R_1x - \frac{wx^2}{2} - W(x-a).$$

The bending moment diagram is drawn as shown in Fig. 116. First consider the weights of the uniform loads on the segments  $a$  and  $b$  as concentrated at their centre points; then draw the diagram  $BDEFA$  as in Case 9; and draw parabolic arcs; one for the portion  $a$ , tangential to  $DE$  and  $DB$  at  $E$  and  $B$ ; the other, for the portion  $b$ , tangential to  $FE$  and  $FA$  at  $E$  and  $A$ .

### 53. Relation between bending moment and shearing force.

#### (a) Concentrated loads.

Let  $AB$  (Fig. 118) be a beam of length  $l$ , supported at  $A$  and  $B$ , and loaded at  $C, D, E, \dots$  with weights  $W_1, W_2, W_3, \dots$ .

Let  $a_1, a_2, a_3, \dots$  be the lengths of the segments  $BC, CD, DE, \dots$ .

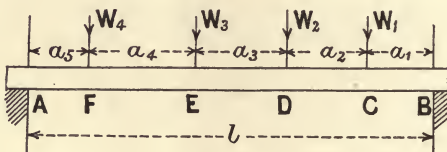


Fig. 118.

The reaction

$$R_1 = \frac{W_1(l - a_1) + W_2(l - a_2 - a_1) + \dots}{l}.$$

The shearing force  $F_1$  between  $B$  and  $C = R_1$ ,

.....  $F_2$  .....  $C \dots D = R_1 - W_1$ ,

.....  $F_3$  .....  $D \dots E = R_1 - W_1 - W_2$ .

.....

The bending moment at  $C = M_C = R_1 a_1 = F_1 a_1$ ,

$$\begin{aligned} \dots \dots \dots D = M_D &= R_1(a_1 + a_2) - W_1 a_2 \\ &= R_1 a_1 + (R_1 - W_1) a_2 \\ &= M_C + F_2 a_2, \end{aligned}$$

$$\begin{aligned} \dots \dots \dots E = M_E &= R_1(a_1 + a_2 + a_3) - W_1(a_2 + a_3) - W_2 a_3 \\ &= M_D + F_3 a_3. \end{aligned}$$

Thus, the difference between the bending moments at the beginning and end of any segment between two consecutive weights, is equal to the shearing force in that segment multiplied by the length of the segment.

(b) Distributed load of uniform intensity, or of continuously varying intensity.

Consider two sections  $KK'$  and  $LL'$  (Fig. 119) of a beam at a very small distance  $dx$  apart.

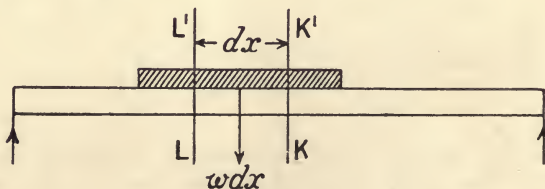


Fig. 119.

Let  $M$  and  $F$  be the bending moment and shearing force at cross section  $KK'$ ;  $M'$  and  $F'$  the bending moment and shearing force at  $LL'$ .

Let  $w$  be the intensity of the load between the sections. Then

$$F - F' = w dx,$$

and taking moments about  $LL'$ ,

$$M' = M + F dx + \frac{w (dx)^2}{2},$$

as  $dx$  is very small  $(dx)^2$  may be neglected.

$$\therefore M' - M = F dx.$$

Let  $dF$  represent the difference between the shearing forces, and  $dM$  the difference between the bending moments; then

$$\frac{dF}{dx} = w \dots \dots \dots (1),$$

$$\frac{dM}{dx} = F \dots \dots \dots (2).$$

From (1) we see, that if a curve of loading be drawn, an ordinate of which at any section gives the load per foot run at that section, then the shearing force  $\left(F = \int_0^x w dx + c\right)$  at any section is equal to the area of the load diagram between the end support and the section.

From (2) it follows, that the shearing force at any section is equal to the rate of change, or increment per unit length of the bending moment; and since to find the section where  $M$  is a maximum, we equate  $\frac{dM}{dx}$  to zero, we see that  $M$  is a maximum where the shearing force changes sign, i.e. where  $F = 0$ .

Again (Fig. 120),

$$F = \frac{dM}{dx} = \tan \theta;$$

hence, in the diagrams of bending moment and shearing force, the ordinate of the shearing force diagram at any point measures the tangent of the slope at the corresponding point of the bending moment curve; also, the difference between the bending moments at any two sections of the span is equal to the area of the shearing force diagram between the same two sections.

Thus, in the case of a uniformly distributed load (Fig. 120):

The bending moment at the centre of span = the area  $acd$  of the shearing force diagram

$$= \frac{1}{2} \cdot \frac{wl}{2} \cdot \frac{l}{2} = \frac{wl^2}{8},$$

and the ordinate  $EG$  of the bending moment diagram is equal to the area of the shaded portion of the shearing force diagram.

It is important to note that when the intensity of the load is constant or varies *continuously*, the equations

$$\frac{dM}{dx} = F, \text{ and } \frac{dF}{dx} = w$$

can be readily integrated as  $w$  and  $F$  are functions of  $x$ ; but in the case of a girder loaded with a series of loads *concentrated at different points* of the span; or with a combined uniform load and concentrated loads; the loading becomes *discontinuous*, and the equations can only be applied between any two consecutive points at which concentrated loads act.

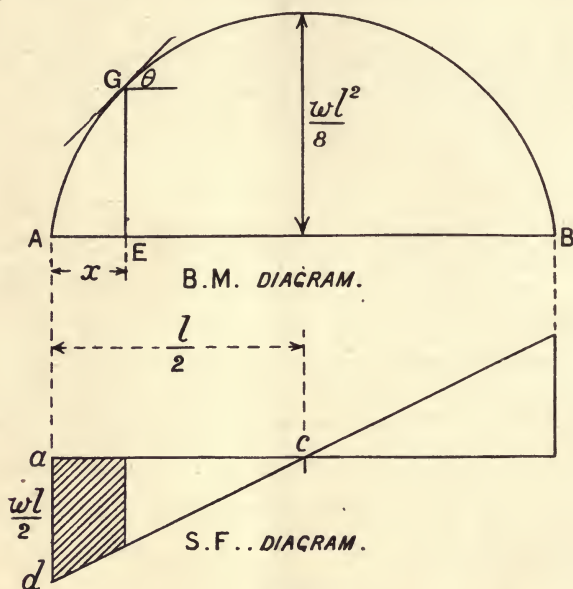


Fig. 120.

54. To show the relation between the polygon formed by a hanging chain, loaded with any system of weights, and the bending moment diagram for a beam similarly loaded.

Let  $AB$  be the beam (Fig. 121), loaded with weights  $W_1, W_2, W_3, W_4$ .

$R_1$  and  $R_2$  the reactions at supports.

For any section  $C$  of the beam, the bending moment

$$M = R_2 l_2 - \sum Wx \dots \dots \dots (1).$$

The force polygon is a vertical line. Take any pole  $O$ , and describe the funicular polygon  $A'C'B'$  (Fig. 122), which is the form a chain would assume if loaded similarly to the beam. Consider the section of chain at  $C'$  vertically below  $C$ . Resolve the stress at  $C'$  into two components, one vertical, the other parallel to  $A'B'$ . The component ( $H$ ) parallel to  $A'B'$  is the same for all sections of the chain. Also resolve the force on the chain at  $B'$  into a vertical component, which must be equal to  $R_2$ , and a component along  $A'B'$  which is equal to  $H$ .

To consider the equilibrium of the portion of chain on right of  $C'$ , take moments about  $E'$ , and let the vertical ordinate

$$E'C' = D.$$

Then

$$-H.D - \Sigma Wx + R_2 l_2 = 0 ;$$

$$\therefore H.D = R_2 l_2 - \Sigma Wx \dots\dots\dots(2).$$

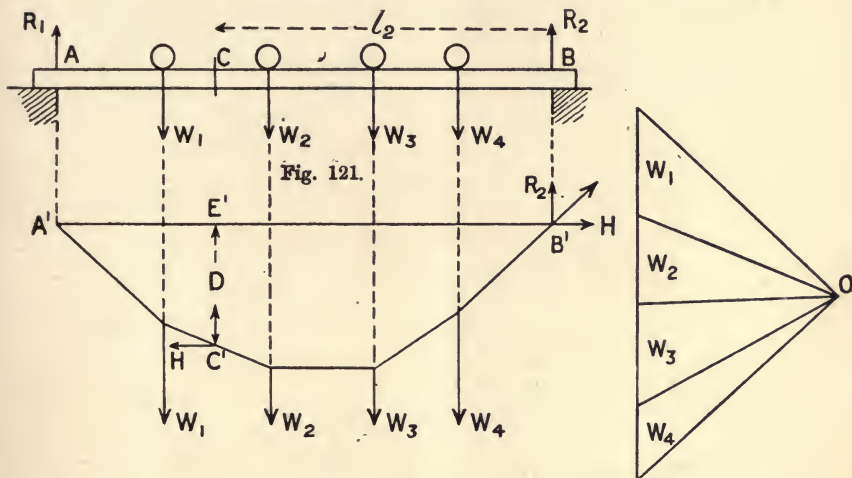


Fig. 122.

Comparing (1) and (2),

$$M = HD.$$

Since  $H$  is constant,  $D$  the ordinate of the funicular polygon is proportional to  $M$ , the bending moment on the beam. Thus the funicular polygon is a diagram of bending moments. If the scale be chosen so that  $H = \text{unity}$ , then

$$M=D.$$

### 55. Examples on bending moments.

*Example I.* A BEAM SUPPORTED AT BOTH ENDS CARRIES A DISTRIBUTED LOAD, VARYING IN INTENSITY AT A UNIFORM RATE FROM ZERO AT ONE END TO  $w$  PER FOOT AT THE OTHER. SKETCH THE CURVES OF BENDING MOMENT AND SHEARING FORCE.

Let  $l = AB =$  length of span.  $ABE$  represents the load.

Let  $W$  = total load (Fig. 123).

Then

$$R_1 l = W \frac{l}{3},$$

and

$$R_1 = \frac{W}{3};$$

$$R_2 = \frac{2}{3} W.$$

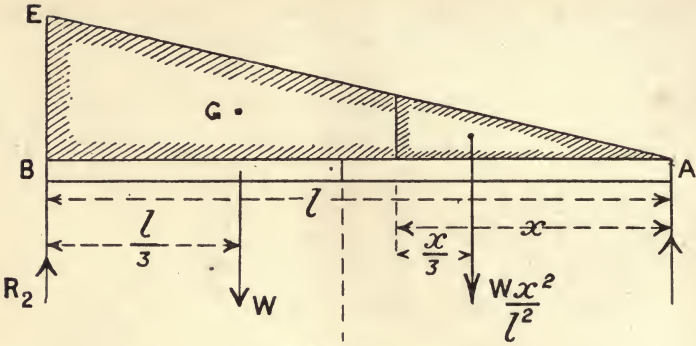


Fig. 123.

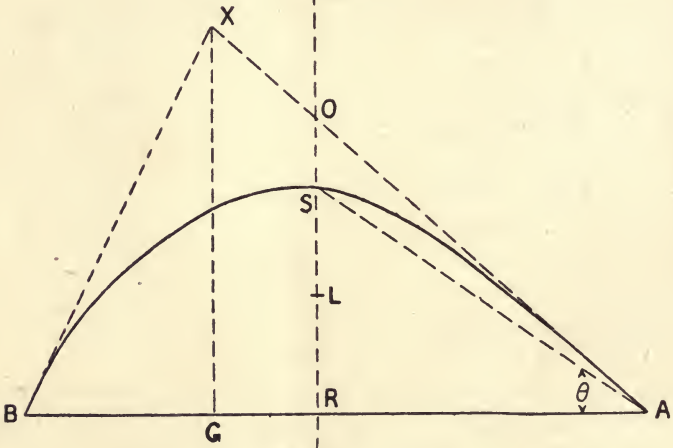


Fig. 124.

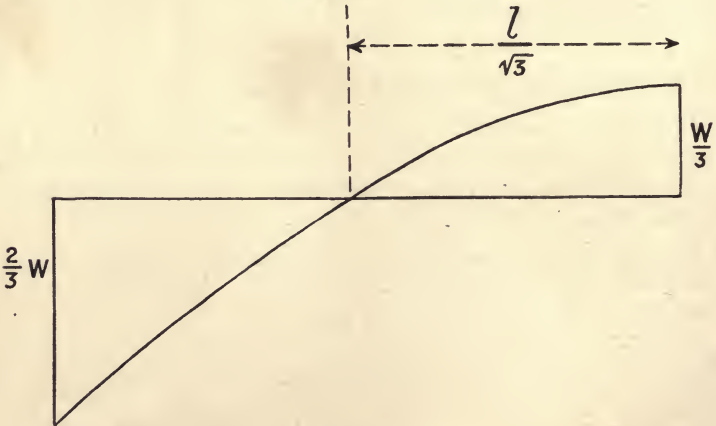


Fig. 125.

Take any section  $D$  distant  $x$  from  $A$ , then total load on  $AD = W \frac{x^2}{l^2}$  acting at distance  $\frac{x}{3}$  from  $D$ ; and

$$M = R_1 x - \frac{W x^2}{l^2} \times \frac{x}{3};$$

$$= \frac{W x}{3} \left( 1 - \frac{x^2}{l^2} \right) \dots \dots \dots (1).$$

To find max.  $M$ ,  $\frac{dM}{dx} = \frac{W}{3} \left( 1 - \frac{3x^2}{l^2} \right) \dots \dots \dots (2).$

This is = 0 when  $x = \frac{l}{\sqrt{3}}$ .

Substituting this value of  $x$  in (1),

$$\text{Max. } M = \frac{W}{3} \frac{l}{\sqrt{3}} \left( 1 - \frac{1}{3} \right) = \frac{2 W l}{9 \sqrt{3}}.$$

To trace the bending-moment curve.

When

$$x = 0, \quad M = 0;$$

$$x = l, \quad M = 0;$$

$$x = \frac{l}{\sqrt{3}}, \quad M = \frac{2 W l}{9 \sqrt{3}}.$$

Take  $AR$  (Fig. 124) =  $\frac{l}{\sqrt{3}}$  and erect the ordinate  $RS = \frac{2 W l}{9 \sqrt{3}}$ , then

$S$  is a point on the curve of bending-moment, and  $\tan \theta = \frac{2 W}{9}$ ; if therefore we bisect  $RS$  in  $L$  and produce  $RS$  to  $O$  making  $SO = SL$ , then the inclination of the line  $AO$

$$\tan OAR = \frac{3 W}{9} \cdot \frac{l}{\sqrt{3}} \div \frac{l}{\sqrt{3}} = \frac{3 W}{9} = \frac{W}{3},$$

which is the inclination of the bending moment curve at  $A$ . If at  $G$  (the centre of gravity of load) we erect a perpendicular to meet  $AO$  at  $X$  and join  $XB$  we get the inclination at  $B$ .

Shearing force at  $D$

$$= R_1 - W \frac{x^2}{l^2};$$

$$= \frac{W}{3} \left( 1 - \frac{3x^2}{l^2} \right) \dots \dots \dots (3)$$

(this equals  $\frac{dM}{dx}$ , see equation 2).

When

$$x = 0, \quad F = R_1 = \frac{W}{3};$$

$$x = l, \quad F = -R_2 = -\frac{2}{3} W;$$

$$x = \frac{l}{\sqrt{3}}, \quad F = 0.$$

Shearing-force curve from equation (3) is evidently a parabola as sketched in Fig. 125.

*Example II.* BEAM HINGED AT ONE END AND OVERHANGING A PIER, LOADED AT THE FREE END, AND BETWEEN THE SUPPORTS.

The reaction  $R$  is got by taking moments about  $A$ .

$$R \times BA = W \times AC + W_1 \times AD.$$

In Fig. 126  $ae$  represents  $W \times AC$ ;

$af$  represents  $W_1 \times AD$ ;

$\therefore ef$  represents  $R \times AB$ .

Join  $fh$  and  $am$ , then  $amhc$  is the diagram of moments.

The bending moment at any section  $K$  is given by the ordinate  $kg$ .

$$\begin{aligned} M \text{ at } K &= R \times BK - W \times CK - W_1 \times DK; \\ &= rs - rk - gs; \\ &= kg. \end{aligned}$$

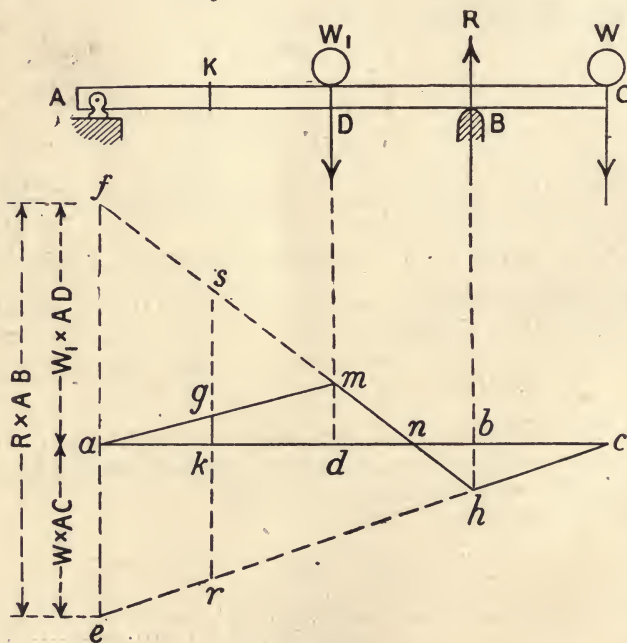


Fig. 126.

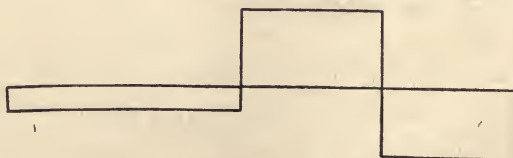


Fig. 127.

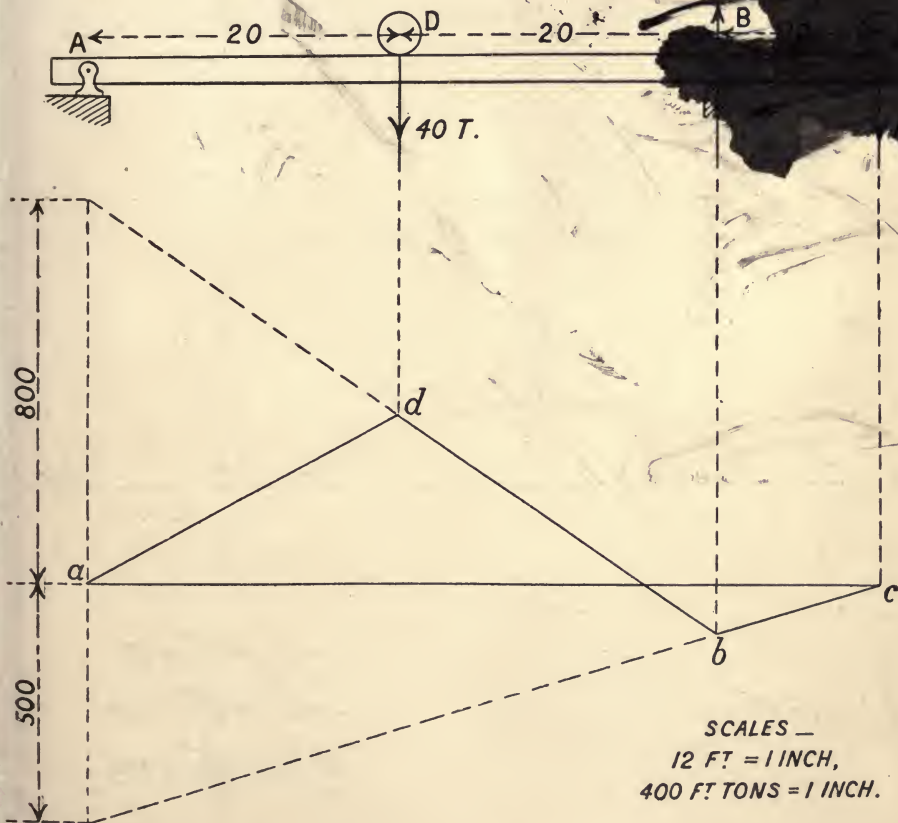
If, as in figure,  $m$  is above  $ad$ , the beam is sagging from  $a$  to  $d$  and hogging from  $n$  to  $c$ .

The point  $n$  where the curvature changes is called a point of contrary flexure.

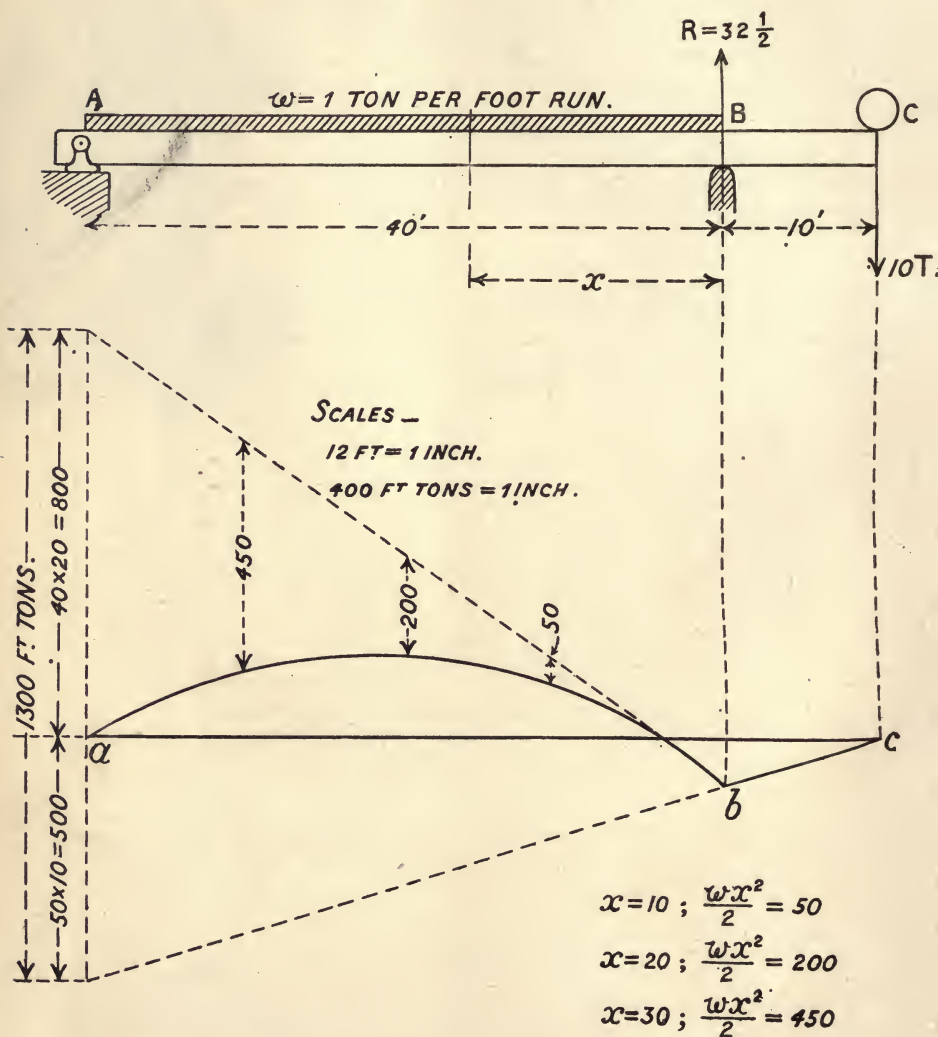
The shearing force diagram is sketched in Fig. 127.

The following numerical examples of overhanging beams are easily understood by the student without further explanation. They are worked on the principles already fully explained. Only the diagrams only are given.

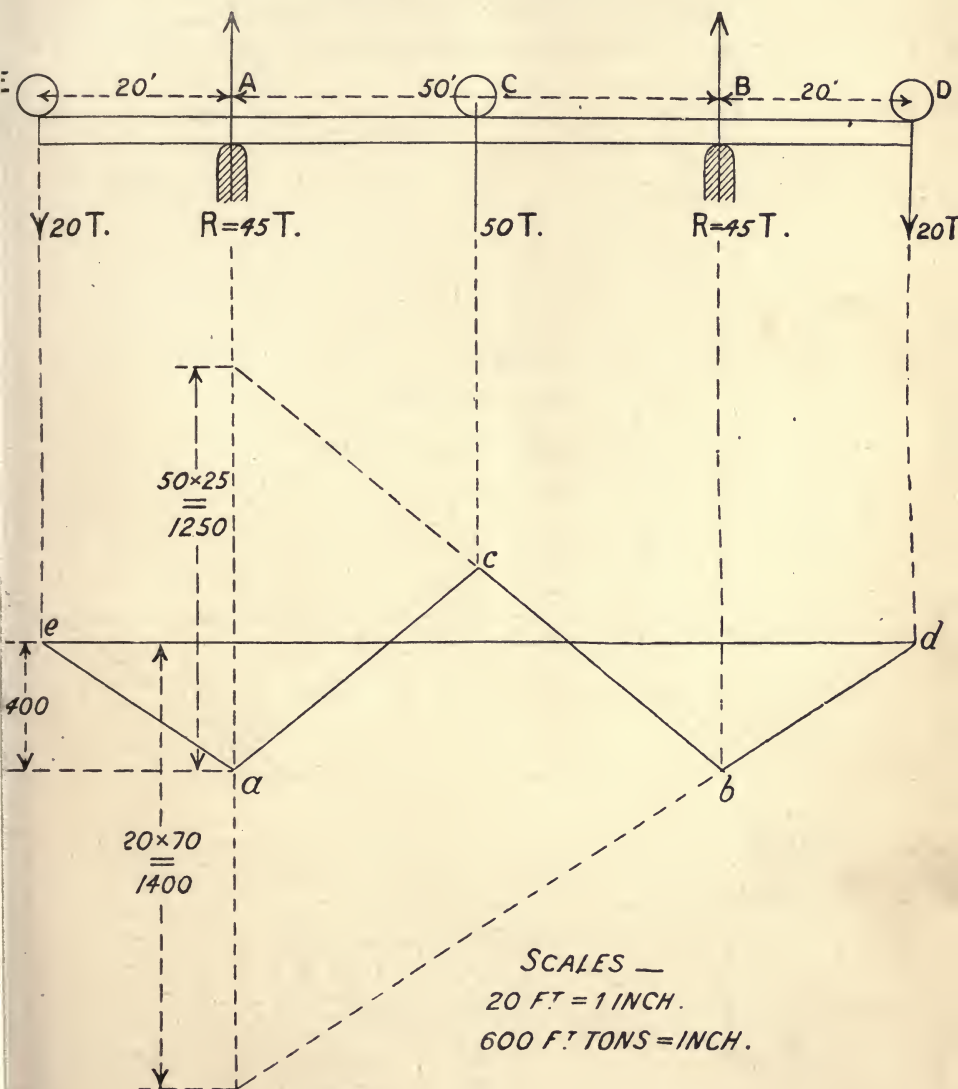
*Example III.* BEAM  $AC$  OF LENGTH 50 FEET, SUPPORTED AT  $A$  AND OVERHANGING THE PIER  $B$ , LOADED WITH 10 TONS PER FOOT AND 40 TONS AT  $D$ .



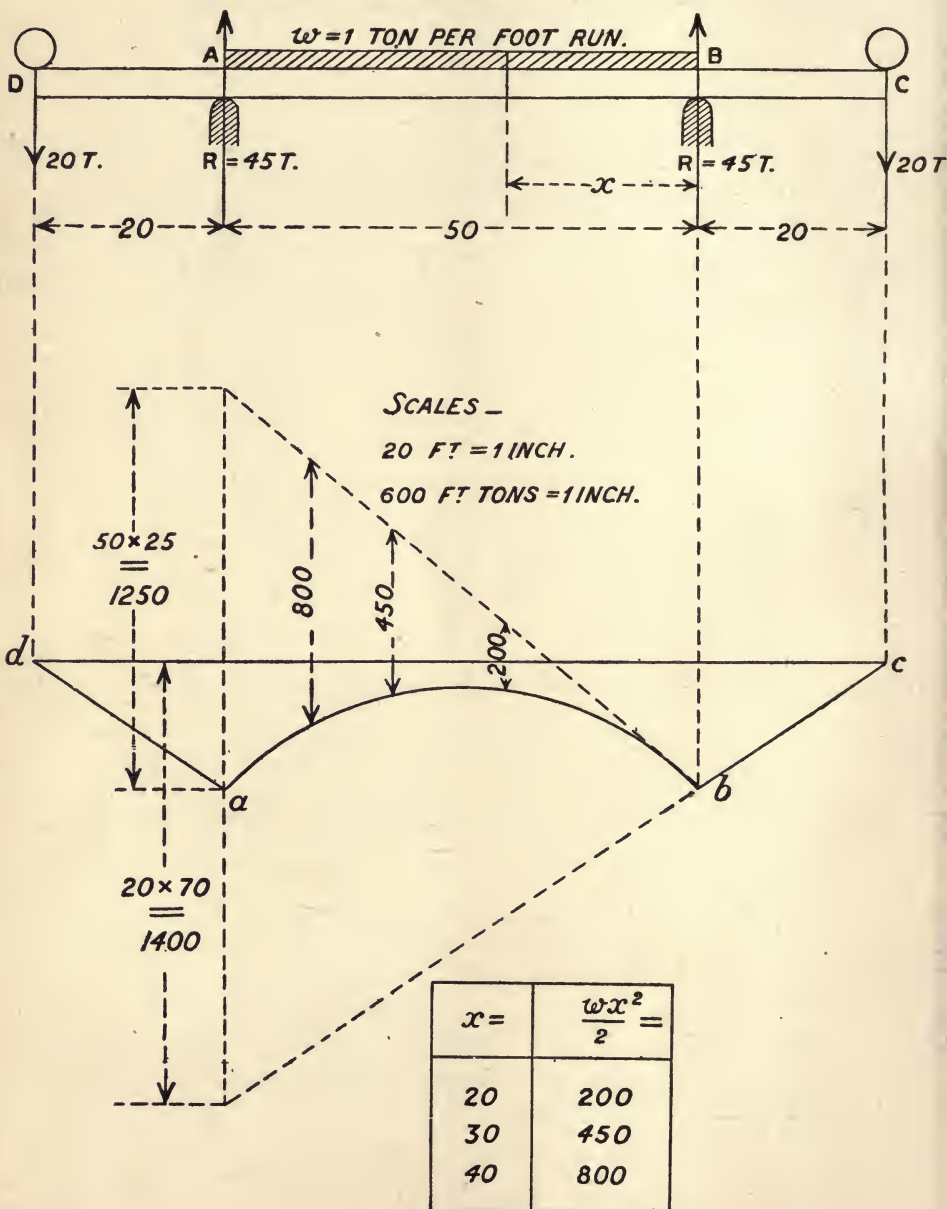
*Example IV.* BEAM  $AC$ , LENGTH 50 FEET, OVERHANGING THE PIER  $B$ , LOADED WITH A WEIGHT OF 10 TONS AT  $C$ , AND A UNIFORM LOAD OF 1 TON PER FOOT BETWEEN  $A$  AND  $B$ .



*Example V.* BEAM OVERHANGING TWO PIERS, LOADED WITH 20 TONS AT EACH END, AND 50 TONS IN CENTRE BETWEEN THE TWO PIERS.



*Example VI.* BEAM, LENGTH 90 FEET, OVERHANGING TWO PIERS, LOADED WITH 20 TONS AT EACH END, AND A UNIFORM LOAD OF 1 TON PER FOOT RUN BETWEEN THE PIERS.



## EXERCISES.

1. A cantilever 20 feet long supports four loads of 5, 6, 7, 8 tons, situated at distances from the fixed end of 20, 16, 10, 5 feet respectively. Find the bending moments at the fixed end, and at a section 12 feet from the fixed end.

*Ans.* 306 foot-tons ; 64 foot-tons.

2. A cantilever 30 feet span carries a uniform load of 2 tons per foot, and, in addition, has a concentrated load of 5 tons at the free end. Find the maximum bending moment, and also that at the centre of span.

*Ans.* Max.  $M = 1050$  foot-tons ;  $M_{\text{centre}} = 300$  foot-tons.

3. A beam 25 feet span supported at both ends carries a uniform load of  $\frac{1}{3}$  ton per foot run distributed over its length ; also two concentrated loads of  $2\frac{1}{2}$  tons at 15 feet and 5 tons at 20 feet from the left support. Draw the curves of shearing force and bending moment. Find the section where bending moment is greatest and its value.

*Ans.* 55 foot-tons at the  $2\frac{1}{2}$  tons weight.

4. A girder 60 feet span, supported at both ends, is loaded uniformly for a distance of 25 feet from the left support, with 2 tons per foot run. Find the bending moments at (a) centre of span ; (b) at end of load ; (c) at centre of uniform load.

Draw curves of bending moments and shearing force.

*Ans.* 312.5 foot-tons ; 364.6 foot-tons ; 338.4 foot-tons.

5. A beam, length 40 feet, supported at both ends, is loaded throughout its length with a uniform load of  $1\frac{1}{2}$  tons per foot run. Find the bending moment and shearing force at  $\frac{1}{8}$ ,  $\frac{1}{4}$ , and  $\frac{1}{2}$  the span from left support.

Draw diagrams of bending moment and shearing force.

*Ans.*  $131\frac{1}{4}$  foot-tons ; 225 foot-tons ; 300 foot-tons ;  $-22\frac{1}{2}$  tons ;  $-15$  tons ; 0.

6. A beam  $AD$ , 40 feet long, is supported at two points  $B$  and  $C$ , so that  $AB = 8$  feet,  $BC = 20$  feet,  $CD = 12$  feet. Weights of 5 tons and 6 tons rest on the extremities  $A$  and  $D$ , and a weight of 12 tons at centre of  $CB$ . Find the bending moments at  $B$  and  $C$ , and at the centre of  $CB$ .

7. If in last example the central portion  $BC$  is loaded with a uniform load of 2 tons per foot run, find bending moments at same places.

8. A beam 50 feet span, supported at both ends, carries concentrated loads of 15 tons, 35 tons, and 20 tons, situated at 5, 20, and 35 feet respectively from the left support. Find the bending moment

at the line of action of each of these weights, and draw curves of bending moment and shearing force.

*Ans.* 202·5 foot-tons ; 585 foot-tons ; 442·5 foot-tons.

9. Draw the shearing-force diagram for a beam 24 feet long, loaded with  $1\frac{1}{2}$  tons per foot, and from it find the bending moment at 6 feet and 12 feet from either support.

10. Two cantilevers, each 50 feet long, placed so that their free ends abut, are loaded with two tons per foot ; show that the bending moment at any section equals 2,500 foot-tons, less the bending moment at the same section for a beam the length of the two cantilevers, loaded in the same way and supported where the cantilevers are fixed.

11. A beam supported at both ends and 35 feet long, is loaded at each intermediate 5 feet with a weight of 2 tons ; draw the bending moment diagram, showing the contribution of each load to the bending moment at every point.

12. A straight prismatic balk of timber floats in a horizontal position with a weight  $W$  at the centre. Draw curves of bending moment and shearing force.

13. A girder, span 120 feet, carries a uniformly distributed load of  $\frac{1}{2}$  ton per foot run, and a central load of 20 tons. Find the bending moment and shearing force at a section 30 feet from either support.

*Ans.* 975 foot-tons ; 25 tons.

14. A beam is placed horizontally upon two supports 14 feet apart, and projects at each end 5 feet beyond the support. A load of 2 tons is placed at the centre of the span, and a load of 3 tons is placed at each of the projecting ends. Calculate the bending moment at the centre and at each support, and sketch the diagram of bending moments.

15. A beam  $ABCDE$  of length 32 feet is divided into 4 equal parts of 8 feet each by the points  $BCD$ . Draw the diagram of bending moments for the following cases :

(a) Beam supported at  $A$  and  $E$ , loaded at  $D$  with 10 tons.

(b) Beam supported at  $D$  and  $B$ , loaded with a uniform load of  $1\frac{1}{2}$  tons per foot from  $B$  to  $D$  and 3 tons at  $A$  and  $E$ .

16. A girder, 30 feet span, supported at the ends, carries a uniform load of 2 tons per foot run extending from one support to the centre of span ; also two loads of 6 tons, one at 8 feet from the left support, the other at 8 feet from the right support. Draw the bending moment and shearing force diagrams.

17. A girder, 40 feet span, supported at the ends, carries a uniform load of  $1\frac{1}{2}$  tons per foot run, which covers the whole span; also two concentrated loads of 6 tons and 10 tons at 8 feet and 24 feet respectively from the right support. Draw the bending moment and shearing force diagrams.

18. For a uniformly loaded beam, show that the area of the load curve is proportional to the ordinate of the curve of shearing force, and the area of the curve of shearing force proportional to the ordinate of the curve of bending moment. Apply this to find the curves of shearing force and bending moment in the case of a beam, the load on which is zero at the ends and increases uniformly towards the middle, proving that the maximum bending moment is  $\frac{1}{8}WL$ .

### 56. Rolling loads.

A *live* or *rolling* load is one which travels over a beam or girder, and occupies different positions at different times. A railway train or traction engine passing over a bridge are examples of a *live* load.

Owing to the load being suddenly applied on a bridge by a train coming on to it with high velocity, and the oscillations set up, and the alterations of stress, which are produced, it is necessary to consider the effect *dynamically* as well as *statically*. At present the static effect only of moving loads will be considered.

CASE 1. BEAM SUPPORTED AT BOTH ENDS, WITH A SINGLE LOAD  $W$  ROLLING ACROSS.

Suppose the load to travel over the beam from  $A$  to  $B$ . Let  $l$  be the length of the span. Consider any section  $K$  distant  $x$  from left support.

BENDING MOMENT. As the load travels from  $A$  towards  $K$  the bending moment at  $K$  is  $R_1(l-x)$ , and as  $R_1$  keeps on increasing, the bending moment increases. After the load passes  $K$ , the bending moment at  $K$  diminishes since its value is now  $R_2 \times x$ , and  $R_2$  diminishes as the load rolls on. *Therefore the maximum value of the bending moment at  $K$  occurs when the load is at  $K$ , and*

$$\begin{aligned}\text{Max. } M_K &= R_1(l-x), \text{ or } R_2x \\ &= W \frac{l-x}{l} x.\end{aligned}$$

The diagram of maximum bending moments is therefore a parabola the ordinate at the centre, where  $x = \frac{l}{2}$ , being  $\frac{Wl}{4}$ . (Fig. 128.)

SHEARING FORCE. The *shearing force* at  $K$  as the load moves towards the section from the left is positive, and equal to  $R_1$ . This

positive shearing force at  $K$  increases with  $R_1$  as the load moves towards  $K$ , and when  $W$  is indefinitely near to  $K$  its value is

$$+\frac{W}{l}x.$$

When the load just passes  $K$  the shearing force becomes negative, and equal to

$$\frac{W}{l}x - W = -\frac{W}{l}(l-x) = -R_2,$$

and as the load moves on towards  $B$  the shearing force at  $K$  diminishes numerically as  $R_2$  diminishes.

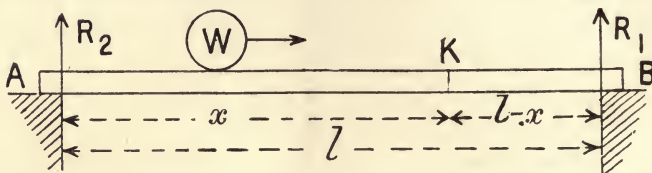


Fig. 128.

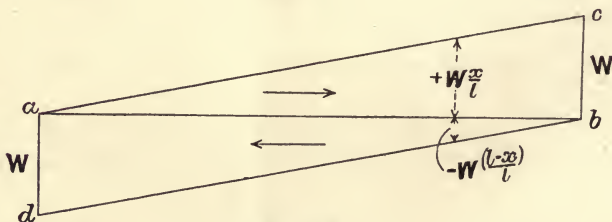


Fig. 129.

Therefore the greatest positive shearing force at any section  $W\frac{x}{l}$  occurs when the load is infinitely near to the section on the left side, and the greatest negative shearing force when the load is infinitely near to the section on the right side, its value being

$$-\frac{W}{l}(l-x).$$

Hence, if  $W$  travels from  $A$  to  $B$ , the successive maximum shearing forces at the several sections of the span are represented by the ordinates of a triangle  $abc$  (Fig. 129), in which  $bc$  is equal to  $W$ , the shearing forces being all positive. If  $W$  travels from  $B$  to  $A$ , the maximum shearing forces are all negative, and are represented by the ordinates of an equal triangle  $abd$ .

CASE 2. TWO CONCENTRATED MOVING LOADS AT A CONSTANT DISTANCE APART.

Let  $AB$  represent the beam of span  $= l$ ,  $W_1$  and  $W_2$  the two loads,  $a$  the distance between them (Fig. 130).

MAXIMUM BENDING MOMENT AT A GIVEN SECTION. The maximum bending moment on a given section occurs when one of the loads is at the section.

First, suppose  $W_1$  at the given section whose distance from  $A$  is  $x$ .

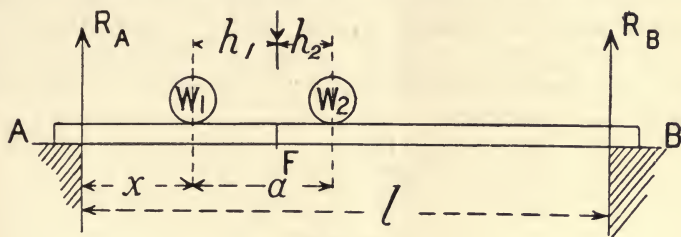


Fig. 130.

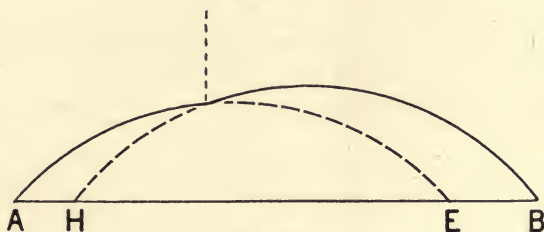


Fig. 131.

Then the reaction

$$R_A = \frac{W_1(l-x) + W_2\{l-(x+a)\}}{l}$$

$$= (W_1 + W_2) \frac{l-x}{l} - W_2 \frac{a}{l},$$

and the bending moment is

$$M_1 = R_A \cdot x = (W_1 + W_2) (l-x) \frac{x}{l} - W_2 a \frac{x}{l} \dots\dots\dots(1).$$

Next, suppose  $W_2$  at the section, with  $W_1$  on the left of it; then

$$R_A = \frac{W_1\{l-(x-a)\} + W_2(l-x)}{l}$$

$$= (W_1 + W_2) \frac{l-x}{l} + W_1 \frac{a}{l},$$

and the bending moment is

$$M_2 = R_A x - W_1 a, \text{ or } R_B (l-x)$$

$$= (W_1 + W_2) (l-x) \frac{x}{l} + W_1 \frac{ax}{l} - W_1 a \dots\dots\dots(2).$$

$$= M_1 + W_2 x \frac{a}{l} - W_1 (l-x) \frac{a}{l}.$$

This last equation shows that  $M_2$  will be greater or less than  $M_1$ , according as  $W_2x$  is greater or less than  $W_1(l-x)$ .

Let  $W_2$  be  $> W_1$ , and let  $AF$  be the value of  $x$  when

$$W_2x = W_1(l-x),$$

so that

$$AF : FB :: W_1 : W_2,$$

then when

$$x > AF, M_2 \text{ is } > M_1,$$

and when

$$x < AF, M_1 \text{ is } > M_2.$$

TO FIND THE POSITION OF THE SECTION WHERE THE GREATEST MAXIMUM BENDING MOMENT OCCURS. First consider the bending moment on any section between  $A$  and  $F$ . When  $W_1$  is at this section, the bending moment is given by equation (1); and differentiating for a maximum

$$\frac{dM_1}{dx} = \frac{W_1 + W_2}{l}(l - 2x) - \frac{W_2a}{l} = 0;$$

$$\therefore x = \frac{l}{2} - \frac{W_2a}{2(W_1 + W_2)} \dots\dots\dots(3).$$

Next consider the bending moment on any section  $x_1$  between  $F$  and  $B$ , measured from  $A$ . When  $W_2$  comes on this section, the bending moment is given by equation (2).

For a maximum differentiate, and

$$x_1 = \frac{l}{2} + \frac{W_1a}{2(W_1 + W_2)} \dots\dots\dots(4);$$

or, if the distances of the maxima bending moments be measured from the centre, they are, for the left portion,  $\frac{W_2a}{2(W_1 + W_2)}$ , and for

the right portion,  $\frac{W_1a}{2(W_1 + W_2)}$ . But the expressions  $\frac{W_2a}{W_1 + W_2}$  and

$\frac{W_1a}{W_1 + W_2}$  represent the distances of  $W_1$  and  $W_2$  respectively from their common centre of gravity, which is in the line of action of their resultant. The absolute maximum bending moment occurs under the heavier load; when this load and the resultant of the two loads are equally distant on opposite sides from the centre of the span.

When the loads are *equal*

$$x = \frac{l}{2} - \frac{Wa}{4W} = \frac{l}{2} - \frac{a}{4}.$$

Hence the position of maximum bending moment will be on either side of the centre, and at a distance  $= \frac{1}{4}a$  from it.

The values of the *greatest maximum* bending moments may be found from equations (1) and (2), by substituting for  $x$  and  $x_1$  the values given in equations (3) and (4). The diagram of bending moments (Fig. 131) will consist of two parabolas intersecting vertically

below  $F$ , one passing through  $A$  and  $E$ , where  $BE$  is the distance of  $W_1$  from centre of gravity of  $W_1$  and  $W_2$ , and the other passing through  $B$  and  $H$ , where  $AH$  is the distance of  $W_2$  from centre of gravity of  $W_1$  and  $W_2$ .

*Example I.* A truck with 3 tons on the left wheel and 7 tons on the right, with wheels 10 feet apart, travels over a beam of 40 feet span. Find the section of the beam which has the greatest maximum bending moment, and its amount.

Here  $W_1 = 3$ ,  $W_2 = 7$ ,  $l = 40$ ,  $a = 10$ .

From equation (3),

$$x = \frac{40}{2} - \frac{7 \times 10}{2(3+7)} = 16.5 \text{ feet.}$$

Substituting this value of  $x$  in (1)

$$M = 10 \times 16\frac{1}{2} \times \frac{23\frac{1}{2}}{40} - \frac{7 \times 10 \times 16\frac{1}{2}}{40} = 69.4 \text{ foot-tons.}$$

From equation (4),

$$x_1 = \frac{40}{2} + \frac{3 \times 10}{2(3+7)} = 21.5 \text{ feet.}$$

Substituting this value for  $x$  in equation (2),

$$M = 10 \times 18\frac{1}{2} \times \frac{21}{40} - \frac{3 \times 10 \times 18\frac{1}{2}}{40} = 83.2 \text{ foot-tons.}$$

Hence, the absolute maximum bending moment occurs in the right half of the beam, at a distance of 1.5 feet from the centre, and its value is 83.2 foot-tons.

*Example II.* A truck with 3 tons on the leading wheel and 7 tons on the back wheel, axles 8 feet apart, crosses a bridge 40 feet span. Find the position and value of the greatest maximum bending moment.

Here  $W_1 = 7$ ,  $W_2 = 3$ ,  $a = 8$ ,  $l = 40$ .

From equation (4),

$$x_1 = 20 + \frac{7 \times 8}{2 \times 10} = 22.8 \text{ feet.}$$

Substituting this value of  $x$  in equation (2),

$$M = \frac{22.8}{40} (172 + 56) - 56 = 73.96 \text{ foot-tons.}$$

From equation (3),

$$x = 20 - \frac{3 \times 8}{2(3+7)} = 18.8 \text{ feet.}$$

Substituting this value of  $x$  in equation (1)

$$M = \{10 \times 21.2 - (3 \times 8)\} \frac{18.8}{40} = 88.36 \text{ foot-tons.}$$

The absolute maximum bending moment occurs in the left half of

the span at a distance of 1·2 feet from the centre, and its value is 88·36 foot-tons.

*Shearing Force.* Consider the loads as in figure,  $W_1$  being the leading load, crossing from  $A$  to  $B$  (Fig. 132).

As the front load  $W_1$  approaches any section  $K$  the positive shearing force at  $K$  increases, since its value is  $R_B$ , and the maximum positive shearing force occurs when  $W_1$  is at an infinitely small distance to the left of  $K$ .

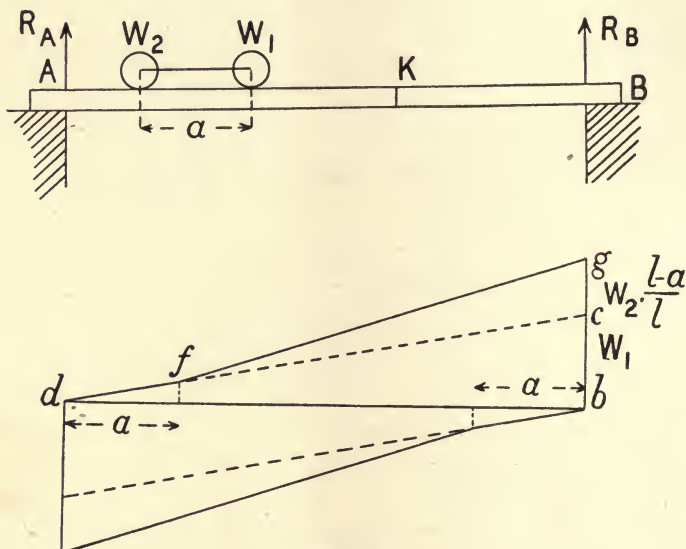


Fig. 132.

When  $W_1$  passes  $K$  this shearing force at  $K$  is diminished by the amount  $W_1$ ; as  $W_1$  moves on from  $K$  towards  $B$  the value of  $+F_K$  increases, on account of the advance of  $W_1$  towards  $B$  and the approach towards  $K$  of the load  $W_2$ , but when  $W_2$  just passes  $K$  there is another diminution in the value of  $+F_K$ , followed by a gradual increase until  $W_2$  reaches  $B$ .

The maximum positive shearing force occurs when the front of  $W_1$  is at  $K$ , and the maximum negative shearing force when the back of  $W_2$  is at  $K$ .

*Diagram of Shearing Force* (Fig. 132). Draw the vertical  $bc$  equal to  $W_1$ ; join  $dc$ ; the ordinates to this line represent the greatest positive shearing force due to  $W_1$  alone.

Produce the vertical  $bc$  to  $g$ , making  $cg = W_2 \frac{l-a}{l}$ , and join  $gf$ , where  $f$  is a point in  $dc$  distant  $a$  from  $d$ .

CASE 3. BEAM SUPPORTED AT BOTH ENDS, CARRYING A MOVING LOAD OF UNIFORM INTENSITY, OF LENGTH GREATER THAN THE SPAN.

The greatest bending moment on a section occurs when the load covers the entire span; for any load placed anywhere on a girder increases the bending moment on any section, and the bending moment is therefore greatest when the whole girder is covered.

The curve of bending moment is a parabola through  $A$  and  $B$ , the ordinate at the centre being  $\frac{Wl}{8}$  or  $\frac{wl^2}{8}$  (Fig. 133).

*Shearing Force.* Suppose the load to travel from left to right; the shearing force at any section  $K$  has its greatest positive value

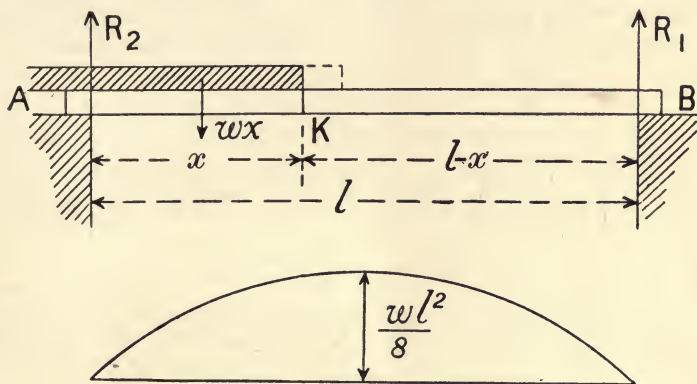


Fig. 133.

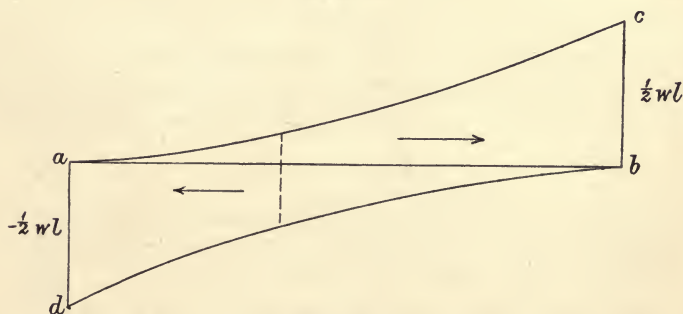


Fig. 134.

when the load covers the portion of the beam lying to the left of the section, and its greatest negative value when the load covers the portion of the beam lying to the right of the section.

When the load occupies the position in figure the shearing force at  $K$  is positive, and equal to  $R_1$ . If the load move on a little so as to pass  $K$ , then  $R_1$  is increased by a *portion* only of the weight so

added, but the downward force to be subtracted from  $R_1$  is the *whole* amount of that added weight, and the resultant, which is the shearing force, is diminished.

When the train covers  $AK$ ,

$$F \text{ at } K = R_1 = \frac{wx^2}{2l} \dots\dots\dots(1).$$

When the train covers  $KB$

$$F \text{ at } K = -R_2 = -\frac{w(l-x)^2}{2l} \dots\dots\dots(2).$$

The shearing force diagram is shown in Fig. 134. The curves are parabolic, the equations of which are given in (1) and (2) respectively.

When the train load *travels from left to right*, the shearing force at the head of train is positive, and is represented by the ordinate from  $ab$  to the parabola  $ac$ , having its vertex at  $a$  and maximum ordinate

$$bc = +\frac{wl}{2}.$$

If the load *travels in the opposite direction* (right to left), the shearing force at the head of train is negative, and is represented by the ordinates of the equal parabola  $bd$ , having its vertex at  $b$  and maximum ordinate

$$ad = -\frac{wl}{2}.$$

The *maximum shearing force* at a given section for any position of the load, occurs when the head of the train (leading axle) is at the section, and the load covers the longer segment of the span.

CASE 4. BEAM SUPPORTED AT BOTH ENDS, LOADED WITH A MOVING LOAD OF LENGTH LESS THAN THE SPAN.

Let  $l$  = length of the span ;  $a$  be the length of the load ;  $b$  = distance of any section  $K$  from the left abutment ;  $x$  = distance from right-hand end of load to section  $K$  ;  $w$  = unit of weight of the load (Fig. 135).

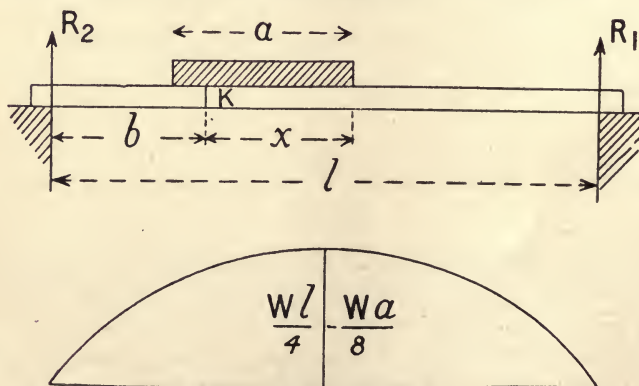


Fig. 135.

To find the Maximum Bending Moment  $K$ .

$$\begin{aligned} M_K &= R_1(l-b) - \frac{wx^2}{2} \\ &= \frac{wa}{l} \left\{ \left( b + x - \frac{a}{2} \right) (l-b) \right\} - \frac{wx^2}{2} \dots\dots\dots(1). \end{aligned}$$

For a maximum  $\frac{dM_K}{dx} = 0$ ;

or  $\frac{wa}{l}(l-b) - wx = 0$ ;

$$x = \frac{a}{l}(l-b);$$

substituting this value of  $x$  in (1),

Maximum bending moment at  $K$

$$\begin{aligned} &= \frac{wa}{l} \left\{ b + \frac{a}{l}(l-b) - \frac{a}{2} \right\} (l-b) - \frac{wa^2(l-b)^2}{2l^2} \\ &= \frac{wa}{l} \left( 1 - \frac{a}{2l} \right) b(l-b). \end{aligned}$$

To find the ordinate at centre put  $b = \frac{l}{2}$  and  $wa = W$ ;

$$M_c = \frac{wa}{8}(2l-a) = \frac{Wl}{4} - \frac{Wa}{8} \dots\dots\dots(2).$$

The curve of maximum bending moment is a parabola with central ordinate

$$\frac{Wl}{4} - \frac{Wa}{8}.$$

*Maximum Shearing Force.* The maximum positive shearing force occurs when the front of the load is at  $K$ , and the maximum negative shearing force when the rear of the train is at  $K$  (Fig. 136). While the load is *only partially* on the girder,

$$\text{Max. } F = + \frac{wx^2}{2l}.$$

When the whole load has just come on,  $x = a$ , and the shearing force  $= \frac{wa^2}{2l}$ . This is the equation to a parabola with vertex at the left end, and maximum ordinate  $= \frac{wa^2}{2l}$ , at a distance  $a$  from the vertex.

When the load comes wholly on the beam, for any position  $x$  measured from  $A$

$$\begin{aligned} \text{Max. } F &= R_1 = \frac{wa}{l} \left( x - \frac{a}{2} \right) \\ &= \frac{W}{l} \left( x - \frac{a}{2} \right); \end{aligned}$$

which is the equation to a straight line of slope  $\frac{W}{l}$ , intersecting  $AB$  at a distance  $\frac{a}{2}$  from  $A$ , and therefore tangential to the parabola.

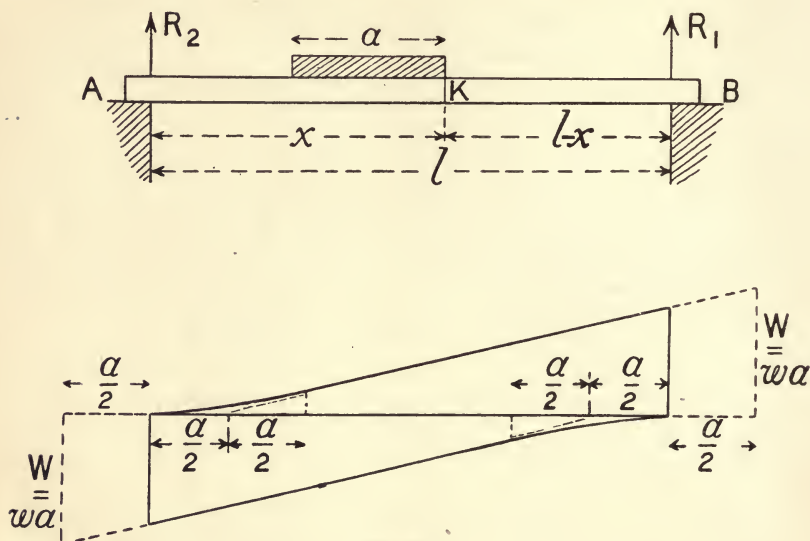


Fig. 136.

A similar curve set out downwards from  $B$  gives the diagram of maximum negative shearing force.

It should be noticed that  $\frac{dM}{dx}$  is *not* equal to  $F$  in the case of moving loads, as is always the case when the load is stationary.

CASE 5. COMBINED DEAD LOAD AND LIVE LOAD BOTH UNIFORMLY DISTRIBUTED.

A girder usually has to support a dead load due to the weight of the structure, as well as a live load moving from one end of the girder to the other; such as a passing train.

*Shearing Force.* The maximum positive and negative shearing forces are got by combining the shearing forces due to each system separately.

Let  $w_1$  be the weight of the dead load per foot run.

Let  $w_2$  be the weight of the live load per foot run.

Let  $l$  be the length of the girder.

In Fig. 137 ordinates to  $ED$  represent the dead load shears, and ordinates to the parabolas  $AJF$ ,  $BKH$  represent maximum live load shears.

For a load travelling from left to right the ordinates of  $ED$  (with their proper sign) are added to the ordinates of  $AJF$  giving the curve

of total shear  $DG$ . Similarly, for a load travelling from right to left we get the curve  $EI$ .

Fig. 138 shows a convenient graphical method of representing the total shear due to dead and live loads, arranged so that the dead load shear is added direct to the live load shear of the same sign.

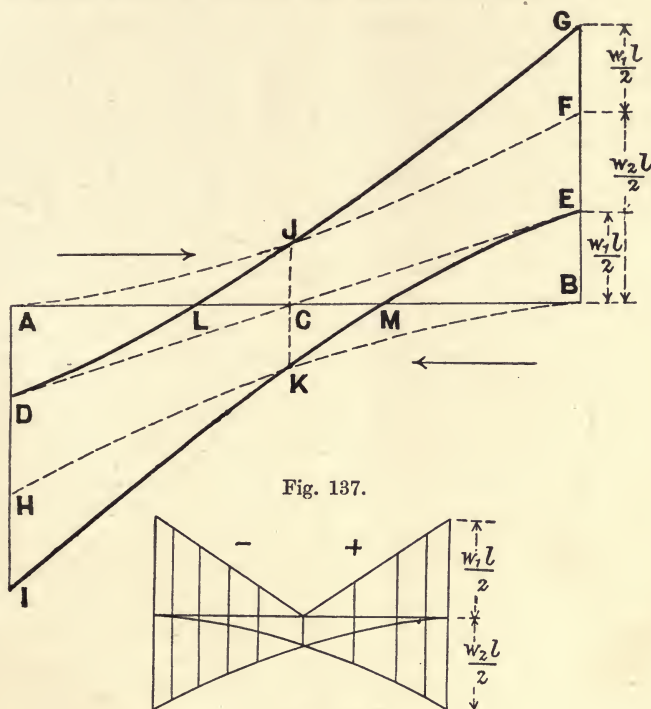


Fig. 137.

Fig. 138.

In Fig. 137 it is seen that the curves  $DG$  and  $EI$  of total shear cross  $AB$  at  $L$  and  $M$ , and for the distance  $LM$  near the centre the shearing force changes sign, according as the train travels over the span from the left, or from the right. This is an important consideration in the case of girders with lattice bracing, as the braces in this central portion  $LM$  of the girder would have to resist either tension or compression. To avoid this reversal of stress in a diagonal brace, a second diagonal brace called a *counterbrace* is introduced in the same panel; one of these only being in action when the shearing force is positive, and the other only when the shearing force is negative.

**Bending Moment.** The maximum bending moment at any section occurs when the girder is fully loaded with both dead and live loads. The curve of maximum bending moment is therefore a parabola with central ordinate =  $\frac{(w_1 + w_2) l^2}{8}$ .

### 57. Bending moments and shearing forces under a system of axle loads.

When the live load is taken as consisting of a series of loaded axles, as in the case of two locomotives followed by a heavy goods train, and it is necessary to find the exact position of the train that will produce the greatest bending moment at any given point in the girder, the simplest way is to solve it graphically. The process is simply sketched out here; the method is fully treated in *Bridge Construction*, by Professor Claxton Fidler, from which this is taken.

The girder over which the train moves is divided into a number of equal parts called bays or panels, and we wish to find the maximum bending moment at each panel point as the train moves. The panel points are 1, 2, 3, 4, 5, 6. It will be seen that it is simpler, for diagram purposes, to make the bridge move forward, the loads remaining stationary.

In Fig. 139 the train is sketched as covering the whole span, the leading axle being directly over the abutment *C*. Denote the successive axle loads by  $W_1, W_2, \dots$ , &c., and their respective distances from the abutment *A* by  $x_1, x_2, \dots$ , &c. The diagram of moments for the downward forces will be the polygonal line *dbcp*, which may be constructed by setting down *ae, eg, \dots*, &c. to represent the moments  $W_1x_1, W_2x_2, \dots$ , &c., as in Case 2, Cantilever with fixed loads. If the bridge were a cantilever fixed at *A*, the bending moments would be represented on this diagram by the vertical ordinates measured below *ad*; but as the girder is supported at *A* and *C*, we have only to superpose the moments due to the supporting force at *C*, which will be represented by the triangle *dap*, and the required moments can be found by measuring the ordinates from the new base *dp* up to the polygonal line.

The polygonal line of moments for the downward forces will serve for any and every position of the train if we regard the abscissæ of the diagram as measured from some point *in the train*, such as the leading axle; for the moment  $W_1x_1$  is simply proportional to  $x_1$ ;  $W_2x_2$  proportional to  $x_2, \dots$ , &c.

Begin, then, by drawing the polygonal line of moments for the downward forces as shown in figure, in which  $x_1$  is equal to the span; and the diagram for any other position of the train will always consist of a portion of the same polygonal line. For the leading axle at *C*, the bending moment at each panel point is found by projecting down the various panel points 1, 2,  $\dots$ , &c., and measuring the ordinate between *pd* and the polygonal line. Thus, at point 2, the bending moment is *os*. Now move the bridge forward one panel length, so that the span becomes  $a_1d_1$ , and leading axle is at 6; the bending moments are then measured on the ordinates under the panel points from the

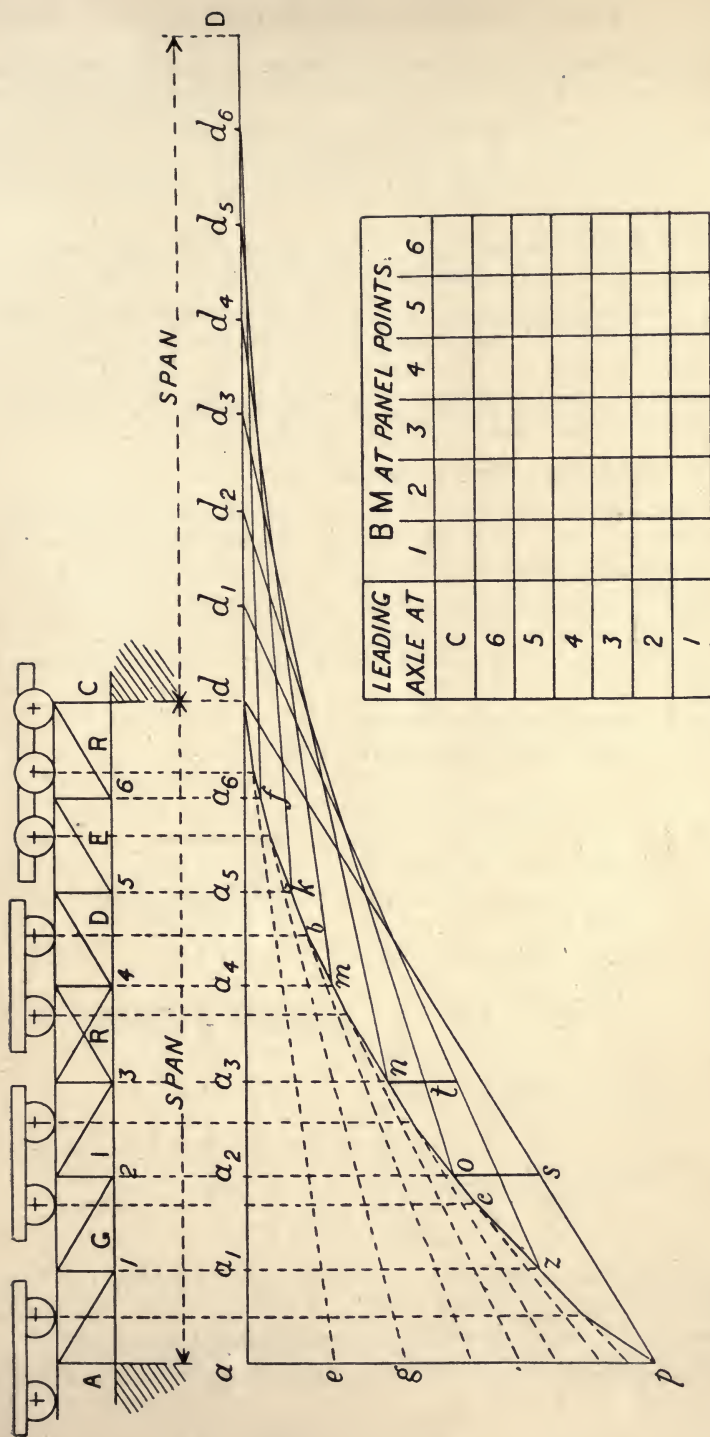


Fig. 139.

line  $d_1z$ . Thus, at panel point 3, leading axle at 6, the bending moment is  $nt$ . Then, having measured all the ordinates, move the bridge forward another panel length, so that the span is now  $d_2a_2$ , and the bending moments are measured from  $d_2o$ , and so on. The results can be tabulated in the form shown, and then the *maximum* moments at each panel taken.

To find the shearing force at any given panel point when the leading axle is any given position to the left of that point, say panel point 6, then the end ordinate  $a_1z$ , scaled off the diagram and divided by the span, will give the shearing force at the point in question, for the shearing force at any section between the leading axle and abutment  $C$  is equal to the reaction at that abutment, and is therefore equal to the moment  $a_1z$  (or  $Wx$ ) divided by the span.

### 58. Shearing stress in beams.

In Fig. 140, let  $CD$  and  $C'D'$  be two vertical sections of the beam separated by a very small distance  $dx$ . It has already been shown that the shearing stress at any point in a vertical section of a beam is accompanied by a shearing stress of equal intensity on a horizontal plane through the point; so that the intensity of shearing stress in the section  $CD$  at  $E$  is equal to the intensity of shearing stress in plane  $EF$ .

Let  $M$  = bending moment at  $CD$ ,

$M + dM$  = bending moment at  $C'D'$ .

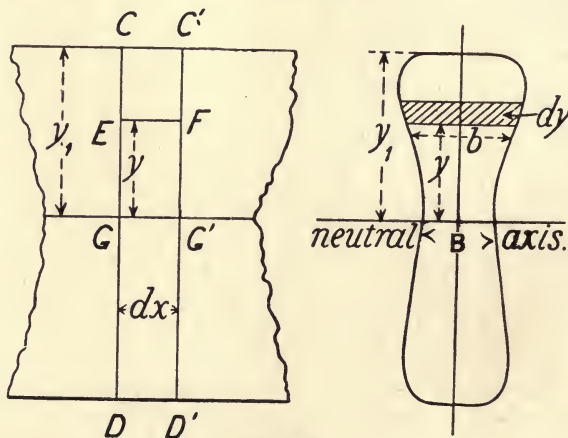


Fig. 140.

Consider the equilibrium of piece of beam  $CEFC'$ ; the forces acting on it are: the normal stresses on  $CE$  due to  $M$ , the normal stresses on  $C'F$  due to  $M + dM$ , and the shearing stress on  $EF$ .

For equilibrium the total shearing stress on  $EF$  must be equal to.

difference between the normal stresses on  $C'F$ , and  $CE$ . We have to state this algebraically.

Let  $q$  = intensity of shearing stress at  $E$ ,

$b$  = width of beam at a height  $y$  from the neutral axis,

$y_1$  = height of top of beam from neutral axis.

Then total shearing stress on  $EF = q \times bdx$  .....(1).

The intensity of the normal stress at  $E$  is  $f = \frac{My}{I}$ , and the difference between this and the corresponding intensity at  $F$  is  $df = \frac{dM}{I} y$ .

Hence the difference between the normal stresses on  $C'F$  and  $CE$

$$= \int_y^{y_1} \frac{dM}{I} y b dy = \frac{dM}{I} \int_y^{y_1} b y dy \text{ .....(2).}$$

Equating (1) and (2)

$$q b dx = \frac{dM}{I} \int_y^{y_1} b y dy;$$

but  $\frac{dM}{dx} = F$ , the shearing force at the section.

$$\text{Therefore } q = \frac{F}{b \cdot I} \int_y^{y_1} b y dy = \frac{F}{bI} A \bar{y} \text{ .....(3),}$$

where  $A$  is the area of the section which extends from  $E$  to  $C$ , and  $\bar{y}$  is the distance of its centre of gravity about the neutral axis. If  $b$  varies we must get it as a function of  $y$ .

Where the lowest limit of the integral in equation (3) is zero, that is, where  $y = 0$ , the shearing stress is a maximum. *The intensity of shearing stress is therefore greatest at the neutral axis, and diminishes to zero at the top and bottom of the section.*

$$\text{Max. } q = \frac{F}{BI} \int_0^{y_1} b y dy, \text{ where } B = \text{width at neutral axis.}$$

In a *rectangular section* let

$b$  = breadth, which is constant,

$d$  = depth; then  $I = \frac{bd^3}{12}$ . (See next chapter.)

The maximum intensity of shearing stress

$$\begin{aligned} &= \frac{F}{I} \int_0^{\frac{d}{2}} y dy = \frac{F}{\frac{bd^3}{12}} \left[ \frac{y^2}{2} \right]_0^{\frac{d}{2}} \\ &= \frac{12F}{bd^3} \frac{d^2}{8} = \frac{3}{2} \frac{F}{bd} = \frac{3}{2} \frac{F}{\text{area of section}}; \end{aligned}$$

that is,  $1\frac{1}{2}$  times the mean intensity over the whole section.

The distribution of the shearing stress may be represented graphically, as in Fig. 141. The shear curve is a parabola as the ordinates vary as  $y^2$ . The intensity of shear at any point of the section  $CD$  is represented by the ordinate at that point to the parabola. The maximum ordinate  $ac$  represents the shearing stress at the neutral axis.

In a *circular section* the maximum intensity of shearing stress is  $\frac{4}{3}$  of the mean intensity.

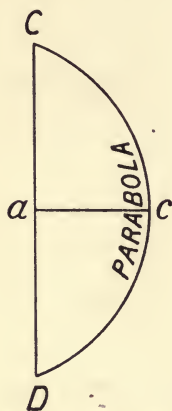


Fig. 141.

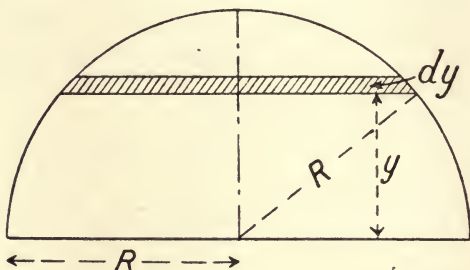


Fig. 142.

Let  $R$  = radius of the circle, then,

$$\text{at } y = 0, \quad b = 2R,$$

$$\text{at } y = y, \quad b = 2\sqrt{R^2 - y^2}.$$

Therefore, as  $I$  for a circle =  $\frac{\pi R^4}{4}$  (see next chapter), maximum intensity of shearing stress

$$\begin{aligned} &= \frac{F}{2R \cdot I} \int_0^R b y dy = \frac{F}{2R} \times \frac{4}{\pi R^4} \int_0^R 2 (R^2 - y^2)^{\frac{1}{2}} y dy \\ &= -\frac{2F}{\pi R^5} \left[ \frac{(R^2 - y^2)^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^R = \frac{4}{3} \frac{F}{\pi R^2}. \end{aligned}$$

In the case of beams of  $I$  section the above equations show that the intensity of shearing stress is greater in the web than in the flanges, and that the distribution over the web is nearly uniform, and is much greater there than in the flanges, as the width  $b$  is so much smaller. In this case a sufficiently accurate result is obtained by assuming the web as bearing the whole shearing force, considered uniform over the section of the web only.

The diagram of shearing stress would be as sketched in Fig. 144, the intensity of shear as shown in Fig. 143.

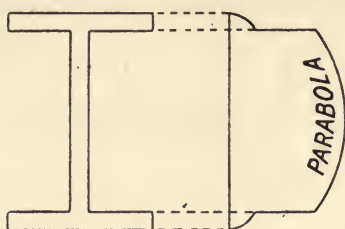


Fig. 143.

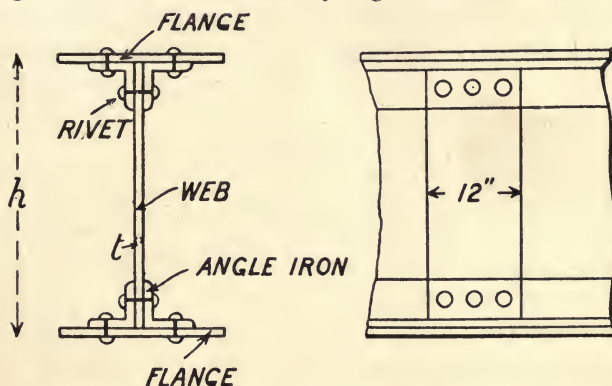


Fig. 144.

When the top flanges are riveted on to the web the rivets and their spacing must be designed to resist the *equal* intensity of *horizontal* shearing stress.

### 59. Riveting of web to flanges.

Fig. 145 gives cross section and elevation showing flanges and web. The flanges and web are connected by angle-irons riveted to them.



CROSS SECTION.

ELEVATION.

Fig. 145.

To find the Rivet area required in one Foot-length to connect the Web and Angle-irons.

If  $F$  = shearing force,

Intensity of shearing stress (in vertical and horizontal directions)

$$= \frac{F}{\text{area of web}} = \frac{F}{t \times h}, \text{ where } t = \text{thickness of web, } h = \text{depth of girder.}$$

Total shear on area of 1 foot-length of web

$$= 1 \text{ foot} \times t \times \frac{F}{t \times h}$$

$$= \frac{F}{h},$$

and this must be equal to the rivet resistance in 1 foot-length, which is equal to the rivet area required multiplied by the working shearing stress.

Therefore

$$\text{Rivet area required in 1 foot} = \frac{\frac{F'}{h}}{\text{working shearing stress}}.$$

*Example.*

A girder 40 feet long, 2 feet deep, is loaded with a uniform load of  $1\frac{1}{2}$  tons per foot run. Thickness of web,  $\frac{3}{8}$  inch. Find the number of  $\frac{3}{4}$ -inch rivets required to connect the web to the flange in 1 foot-length at the supported ends.

Assume the working shearing stress for rivets, 5 tons per square inch.

The shearing force at each end is  $\frac{40 \times 1\frac{1}{2}}{2} = 30$  tons.

Cross-sectional area of web =  $24 \times \frac{3}{8} = 9$  square inches.

Intensity of shearing stress at ends =  $\frac{30}{9} = 3\frac{1}{3}$  tons per square inch.

The area of web for 1 foot = 12 inches in length =  $12 \times \frac{3}{8} = 4\frac{1}{2}$  square inches.

Therefore *total shear* in 1 foot-length of web at ends =  $4\frac{1}{2} \times 3\frac{1}{3} = 15$  tons.

As there are two angle-irons, each rivet connecting web to angle-irons has two sections to resist the shear, so that *each* rivet has a resistance to shear

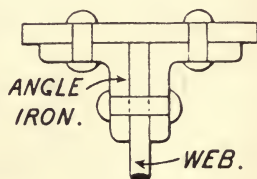
$$= 2 \times \frac{2\frac{1}{2}}{7} \times \left(\frac{3}{8}\right)^2 \times 5.$$

Therefore number of rivets required

$$= \frac{15}{2 \times \frac{2\frac{1}{2}}{7} \times \left(\frac{3}{8}\right)^2 \times 5} = \frac{33\frac{6}{9}}{5} = 3\cdot4, \text{ say } 4.$$

Or four rivets per foot are required to connect the angle-irons to web. The pitch (distance centre to centre) equals 3 ins.

As the rivets connecting the angle-irons to flange-plate are in *single* shear, the number required per foot will be eight.



## EXERCISES.

1. A rolling load of  $1\frac{1}{2}$  tons per foot run travels over a bridge of 140 feet span. Find the maximum bending moments and shearing forces which can occur at sections of the bridge distant 40, 70, and 100 feet respectively from the left support.

Sketch the diagrams of maximum bending moments and shearing forces for the whole bridge due to this rolling load.

*Ans.* 3000 ; 3575 ; 3000 foot-tons.

53·6 ; 26·25 ; 8·6 tons.

2. A four-wheeled truck weighing 8 tons travels over a girder 40 feet span ; 5 tons resting on the leading pair of wheels, and 3 tons on the trailing wheels ; the axles are 8 feet apart. Find (a) the maximum bending moment on a section 12 feet from the right support, (b) the position and value of the maximum possible bending moment.

Sketch with dimensions the diagrams of maximum bending moment and shearing force for the whole span due to the above travelling load.

3. Prove that when a bridge is liable to be covered wholly or partly by a uniform advancing load the bending moment at any section is greatest when the bridge is wholly covered, and the shearing force at any section has its greatest positive and negative values when the load extends from the section to one or other of the piers.

4. A girder 40 feet span is subject to a travelling load of 3 tons per foot run, of length not less than the span, also a uniform dead load of 1 ton per foot. Find the maximum positive and negative shearing forces at intervals of 5 feet for the combined loads.

Sketch the shearing force diagram for the whole girder.

5. The shearing force at a section of a plate girder is 120 tons. Its depth at that section is 6 feet. Find (a) the thickness of the web-plate, (b) the number of rivets required per foot for uniting the web-plate to the booms. Working shearing stress = 10,000 lbs. per sq. inch. Diameter of rivets 1 inch.

6. A horizontal beam supported at its ends *A* and *B* is traversed by a moving load *W* uniformly distributed over a segment *PQ* of constant length ; show that the bending moment at any point *K* of the beam is greatest when *K* divides *PQ* in the same ratio as that in which it divides *AB* ; and show that this maximum bending moment is

$$\frac{W}{AB^2} AK \cdot KB (AB - \frac{1}{2} PQ).$$

7. A girder of length  $l$ , supported at both ends, carries a fixed load of  $w_1$  tons per foot run, also a moving load of  $w_2$  tons per foot run, travelling from one end of the span to the other, and finally covering the whole span. Find the maximum bending moment and shearing force on any given section.

Trace the curves of maximum shearing force due to both loads, the moving load travelling in either direction over the girder.

8. A travelling load of  $1\frac{1}{2}$  tons per foot run, of length greater than the span, passes over a girder of 100 ft. span. Find the maximum bending moment and shearing force at a section 40 feet from the right support.

*Ans.* 1800 foot-tons. 27 tons.

9. If the girder in last question also supports a dead load of 1 ton per foot run, find the maximum bending moment and shearing force at the same section due to both loads.

*Ans.* 3000 foot-tons. 37 tons.

10. A plate girder 60 feet span, depth of web 5 feet, carries a uniform load of 4 tons per foot run. Find (a) the thickness of the web at end of girder, (b) pitch of the rivets connecting web and flanges. Diameter of rivets  $\frac{7}{8}$  inch. Working stress = 4 tons per sq. inch.

*Ans.*  $\frac{1}{2}$  inch; 5 rivets per foot.

11. A bridge girder 84 feet span carries two equal concentrated loads, each 16.4 tons travelling over the span. The loads are fixed at a constant distance apart of 12 feet. Find the section on which the absolute maximum bending moment occurs, and the value of that bending moment.

*Ans.* At  $x = 39$  feet; max.  $M = 593.2$  foot-tons.

12. A girder with parallel flanges 150 ft. span, is divided into 10 panels each 15 ft. long. A live load which covers the whole span, consisting of two engines and trucks, passes over the bridge. The loads in tons on one girder counting from the fore end of train are 9; 9; 9; 5; 5; 5; 9; 9; 9; 5; 5; 5; 5; 5; 5; 5; 5; 5; 5; at distances apart starting from the same end of 11; 11; 9; 7; 7; 9; 11; 11; 9; 7; 7; 8; 10; 8; 10; 8; 10 ft. Determine graphically the maximum bending moment at each panel point.

## CHAPTER VI.

### MOMENTS OF INERTIA.

#### 60. Moments of area and moments of inertia.

DEFINITIONS. **MOMENT OF AREA.** If we suppose a surface divided up into a number of small elementary areas, then the "*moment of area*" about any line in its plane as axis is the sum of the products of each elemental area into its perpendicular distance from that axis; the perpendicular distances which lie on one side of the axis being reckoned positive, and those on the other side negative. As in problems we have usually to deal with, the axis is horizontal, we will consider distances measured *up* as positive, and when measured *down* as negative.

An axis about which the moment of area is *zero* passes through a point called the "*centre of area*," or more usually the "*centre of gravity*" of the surface. As the neutral axis passes through the centre of gravity of the cross section, it is important to state how in unsymmetrical sections the centre of gravity can be readily obtained.

Let  $A$  = whole area of surface, and suppose it divided up into elements of area  $a_1, a_2, a_3, \dots$ , &c., the distances of whose centres of area from any plane are  $y_1, y_2, y_3, \dots$ , &c., and let  $\bar{y}$  be the distance of the centre of area of the whole surface from the same plane, then

$$\bar{y} = \frac{a_1 y_1 + a_2 y_2 + a_3 y_3 + \dots}{A}$$

or

$$A\bar{y} = \int ay.$$

The **MOMENT OF INERTIA** of a surface, about a line in its plane as axis, is the sum of the products of each elementary area into the square of its distance from the axis.

A moment of inertia is always positive, being the product of the square of a length into an area. It is usually denoted by the symbol  $I$ , and

$$\begin{aligned} I &= a_1 y_1^2 + a_2 y_2^2 + a_3 y_3^2 + \dots \\ &= \int ay^2 \\ &= Ak^2 \end{aligned}$$

where  $A$  is the whole area, and  $k$  the radius of gyration of the *area* about the axis.

If a surface is divided up into several portions, the moment of inertia of the whole surface about any axis is equal to the sum of the moments of inertia of the several portions about the same axis. If a section is conceived as the difference of two figures, the moment of inertia of the section about any axis is equal to the difference of the moments of inertia of the two figures about the same axis.

### 61. Principal axes and principal moments of inertia of a plane surface.

The rectangular axes intersecting at the centre of area of a given surface, round which the moments of inertia are respectively *greatest* and *least*, are called the principal axes. It is important, especially in the case of struts, to know the principal moments of inertia of an area, as the resistance of a piece to flexure will be greatest and least round those axes of greatest and least moment of inertia respectively.

If a surface has *one axis of symmetry*, that axis traverses the centre of area, and is one of the principal axes, the other principal axis being perpendicular to it.

If a surface has only *two axes of symmetry at right angles*, they are the principal axes of that surface, and their intersection is the centre of area.

If the moments of inertia about the two principal axes are equal, then every axis through the centre of area is a principal axis, and the moments of inertia are the same about each axis; and if a surface has *two axes of symmetry not at right angles*,—for example, an equilateral triangle,—the moment of inertia about each and every axis through their intersection (the centre of area) is the same, every axis being a principal axis. This is the case with circular and all regular polygonal surfaces.

### 62. Moment of inertia of a plane area about an axis perpendicular to the area.

Let Fig. 146 represent a plane area. Take  $O$  any point, and draw  $OX$ ,  $OY$  through it at right angles; also assume a third axis  $OZ$ , perpendicular to  $OX$ ,  $OY$ , and therefore to the plane of the surface. Let  $N$  be a small element of area " $a$ ," the coordinates of which are  $x$  and  $y$ . The moment of inertia of the element  $N$  round the axis  $OZ$  is  $a\overline{ON}^2$ . Therefore the moment of inertia of the whole surface round that axis is

$$\begin{aligned} I_{OZ} &= \int a\overline{ON}^2 = \int a(x^2 + y^2) = \int ax^2 + \int ay^2 \\ &= I_{OX} + I_{OY}. \end{aligned}$$

$I_{OZ}$  is called the *polar moment of inertia*, and is usually designated by the letter  $J$ .

Hence the moment of inertia of an area about an axis  $O$  at *right angles* to the area is equal to the *sum* of the moments of inertia about

any two axes at right angles to one another, through the point  $O$ , and in the plane.

### 63. Moments of inertia about parallel axes.

The following theorem is frequently necessary in determining the moments of inertia of beams of complex sections :

*The moment of inertia of an area with respect to any axis in its plane is equal to the moment of inertia of the area about a parallel axis passing through its centre of gravity, plus the product of the area into the square of the distance between the two axes.*

In Fig. 147 let  $A$  be an area of any shape,  $XOX$  its neutral axis traversing the centre of area,  $SS$  any axis in the plane, parallel to  $XOX$

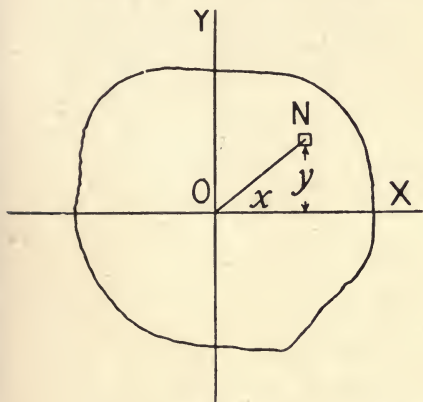


Fig. 146.

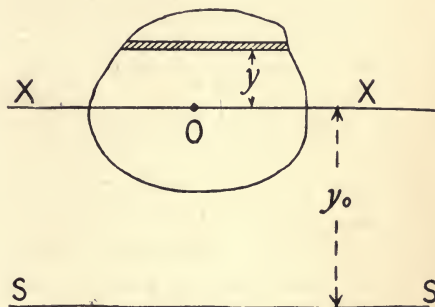


Fig. 147.

and distant  $y_0$  from it. Take a small elemental strip of area " $a$ " parallel to  $XOX$  and distant  $y$  from it.

The moment of inertia of this strip relatively to  $SS$  is  $a(y + y_0)^2$ , and of the whole surface is

$$I_{SS} = \int a(y + y_0)^2 = \int a(y^2 + 2yy_0 + y_0^2),$$

or, since  $y_0$  is constant,

$$\begin{aligned} &= \int ay^2 + 2y_0 \int ay + y_0^2 \int a \\ &= I_{XX} + Ay_0^2; \end{aligned}$$

the second term being zero, since  $XOX$  passes through the centre of gravity of  $A$ , and therefore  $\int ay = 0$ .

### 64. Moments of inertia of plane areas.

#### CASE 1. THE SURFACE A RECTANGLE.

Let Fig. 148 represent a rectangular section, of which  $b$  is the breadth and  $d$  the depth. The neutral axis is parallel to  $b$  and

bisects the depth  $d$  of the rectangle. Take a small strip of area  $b dy$  at a distance  $y$  from the neutral axis. Then, owing to the symmetry of the section above and below the neutral axis,

$$I_{XX} = 2 \int_0^{\frac{d}{2}} b dy y^2 = 2 \left[ \frac{b y^3}{3} \right]_0^{\frac{d}{2}} \\ = \frac{bd^3}{12} = A \frac{d^2}{12} = A \frac{\left(\frac{d}{2}\right)^2}{3},$$

where  $A$  = area of rectangle.

Similarly

$$I_{YY} = \frac{db^3}{12} = A \frac{b^2}{12},$$

and the radii of gyration

$$k_{XX} = \frac{d}{2\sqrt{3}}; \quad k_{YY} = \frac{b}{2\sqrt{3}}.$$

The moment of inertia about  $AB$

$$I_{AB} = \frac{bd^3}{12} + bd \left(\frac{d}{2}\right)^2 = \frac{bd^3}{3} = A \frac{d^2}{3}.$$

#### CASE 2. RECTANGLE SYMMETRICALLY HOLLOWED.

Figs. 149 and 150 illustrate two varieties of this form; the one hollowed internally, the other with equal and symmetrical hollows.

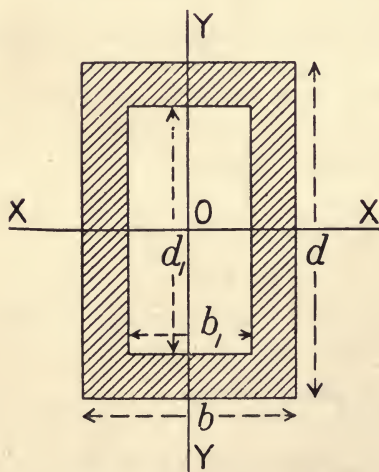


Fig. 149.

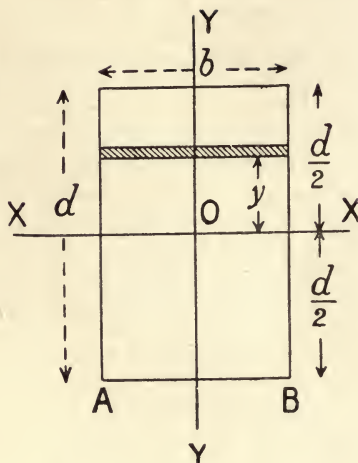


Fig. 148.

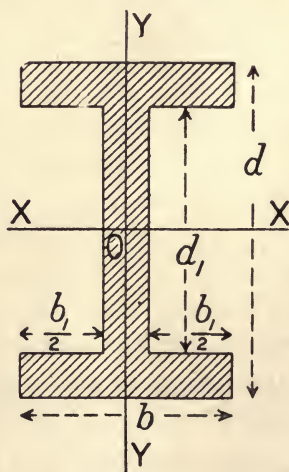


Fig. 150.

The moment of inertia is the moment of inertia of the exterior rectangle less that of hollows, and therefore

$$I_{XX} = \frac{1}{12} (bd^3 - b_1d_1^3),$$

and

$$k_{XX} = \frac{1}{2\sqrt{3}} \sqrt{\frac{bd^3 - b_1d_1^3}{bd - b_1d_1}}.$$

### CASE 3. CIRCLE ABOUT A DIAMETER.

This is most easily obtained from the polar moment of inertia. All axes through the centre  $O$  of the circle are principal axes, and the moments of inertia about those axes are all equal.

It has been shown that the moment of inertia about an axis through  $O$  at right angles to the area is the sum of the moments of inertia about two rectangular axes through  $O$  in the plane of the area.

Or

$$J = I_{OX} + I_{OY}.$$

But in the case of a circle  $I_{OX} = I_{OY}$ .

Therefore

$$J = 2I_{OX} = 2I_{OY}.$$

Let  $R$  = radius of the circle (Fig. 151).

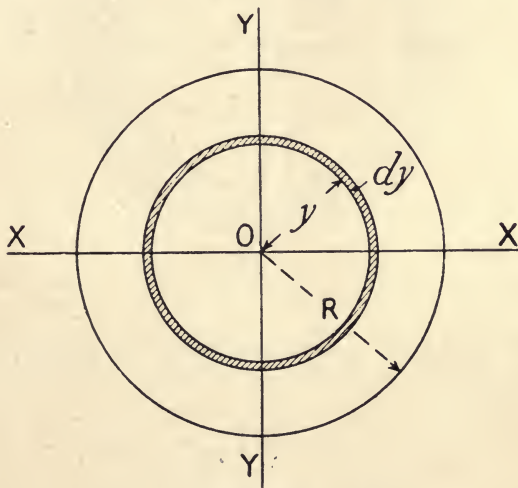


Fig. 151.

To find  $J$ , conceive the circular surface divided into concentric elements of area.

Let  $dy$  be the thickness of one of these, whose distance from  $O$  is  $y$ .

$$\text{Then } J = \int_0^R 2\pi y dy \times y^2 = 2\pi \int_0^R y^3 dy = \frac{\pi R^4}{2} = \frac{\pi D^4}{32}.$$

Therefore moment of inertia about *any diameter*

$$I = \frac{\pi R^4}{4} = \frac{\pi D^4}{64};$$

and

$$k = \frac{1}{2} R = \frac{D}{4}.$$

The moment of inertia about a *tangent line* is

$$= \frac{\pi R^4}{4} + \pi R^2 \times R^2 = \frac{5}{4} \pi R^4 = \frac{5}{64} \pi D^4.$$

#### CASE 4. CIRCULAR RING.

Let  $R_1$  and  $R_2$  be the external and internal radii of the ring (Fig. 152).

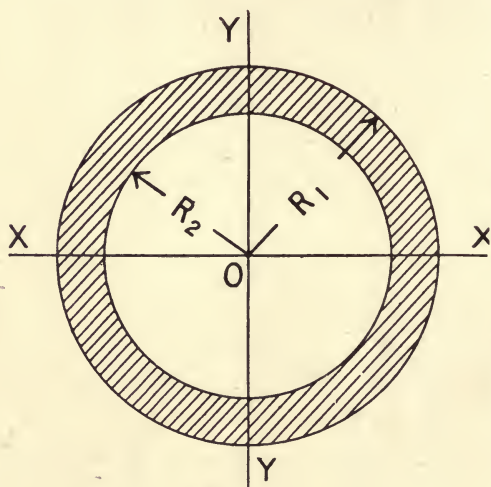


Fig. 152.

Then the moment of inertia and radius of gyration round any diameter are equal to the difference of those of the outer and inner circles. That is

$$I = \frac{1}{4} \pi (R_1^4 - R_2^4) = \frac{\pi}{64} (D_1^4 - D_2^4);$$

$$k = \frac{1}{2} \sqrt{R_1^2 + R_2^2} = \frac{1}{4} \sqrt{(D_1^2 + D_2^2)}.$$

#### CASE 5. MOMENT OF INERTIA OF A TRIANGLE ABOUT AN AXIS THROUGH THE CENTRE OF AREA PARALLEL TO THE BASE.

In Fig. 153 let  $b$  be the base, and  $h$  the height of triangle. Then centre of area is at a height  $\frac{h}{3}$  above the base.

First find the moment of inertia about  $AB$  as axis. Take a small

strip of area parallel to, and at a distance  $y$  from  $AB$ . The area of this strip is  $xdy$  where  $x = \frac{b}{h}(h-y)$ .

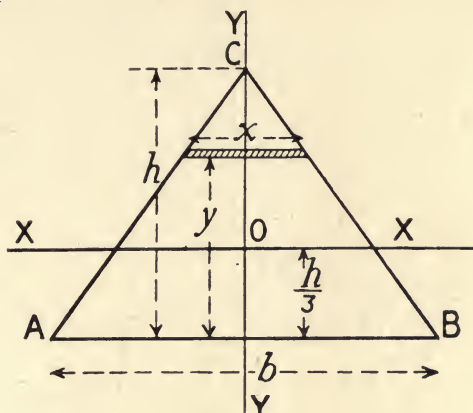


Fig. 153.

$$\begin{aligned} \text{Therefore } I_{AB} &= \frac{b}{h} \int_0^h (h-y) y^2 dy = \frac{b}{h} \left( \frac{h^4}{3} - \frac{h^4}{4} \right) \\ &= \frac{bh^3}{12}; \end{aligned}$$

and

$$\begin{aligned} I_{XX} &= \frac{bh^3}{12} - A \left( \frac{h}{3} \right)^2 = \frac{bh^3}{12} - \frac{bh^3}{18} \\ &= \frac{bh^3}{36}. \end{aligned}$$

CASE 6. MOMENT OF INERTIA OF TEE SECTION IN FIG. 154 ABOUT AXIS  $XX$ .

It is first necessary to find the centre of area, and consequently the position of neutral axis.  $O$  is centre of area of whole surface,  $O_1$  and  $O_2$  the centres of area of web and flange. Take moments of area about the lower edge of flange (Fig. 154),

$$\begin{aligned} \bar{y} &= \frac{4 \times \frac{1}{2} \times 2\frac{1}{2} + 2 \times \frac{1}{2} \times \frac{1}{4}}{2 + 1} \\ &= 1.75 \text{ inches.} \end{aligned}$$

The distance of centre of web from neutral axis = 0.75 inches.  
The distance of centre of flange from neutral axis = 1.5 inches.  
Therefore

$$\begin{aligned} I_{XX} &= \overbrace{\frac{1}{2} \times \frac{4^3}{12} + \frac{1}{2} \times 4 \times 0.75^2}^{\text{web}} + \overbrace{\frac{2 \times \frac{1^3}{2}}{12} + 2 \times \frac{1}{2} \times 1.5^2}^{\text{flange}} \\ &= 2.66 + 1.12 + 0.02 + 2.25 = 6.05 \text{ inch units.} \end{aligned}$$

This example has been shown in detail, but the most systematic method is to tabulate the results, as explained in the following article and examples.

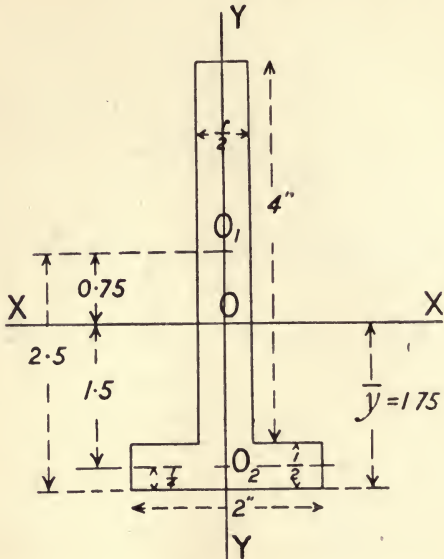


Fig. 154.

**65. Moment of inertia of section composed of rectangles.**

When the section is made up of a series of rectangles the moment of inertia can be found as follows. First find the position of neutral axis by taking moments of area about some fixed line in the plane, and

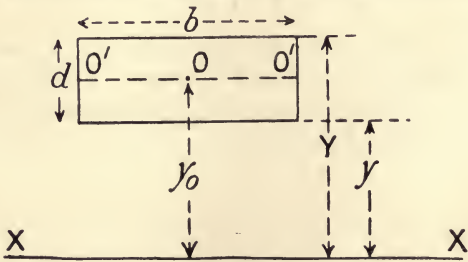


Fig. 155.

then dimension on the section for each rectangle the distance y<sub>0</sub> of its centre of gravity from the neutral axis. Then, for each rectangle (Fig. 155)

$$\begin{aligned} I_{XX} &= I_{O'O} + Ay_0^2, \\ &= \frac{bd^3}{12} + Ay_0^2 = A \left( \frac{d^2}{12} + y_0^2 \right) \dots\dots\dots(1). \end{aligned}$$

The total area is the sum of the areas of the several rectangles; the total moment of area about  $XX$  is the sum of the partial moments; and the total moment of inertia about  $XX$  is the sum of the partial moments of inertia about this axis.

The same result can be obtained in another form, by using the values of  $Y$  and  $y$ , the ordinates to the top and bottom edges of the rectangles, instead of  $y_0$ .

The rectangle (Fig. 155) may be taken as the difference of two rectangles, whose sides are  $b$ ,  $Y$ , and  $b$ ,  $y$ .

Thus *moment of area* about the axis  $XX$  is

$$= bY \times \frac{Y}{2} - by \frac{y}{2} = \frac{b}{2} (Y^2 - y^2) \dots\dots\dots (2).$$

Its *moment of inertia* about  $XX$  (the base), Case I., is

$$I_{XX} = \frac{bY^3}{3} - \frac{by^3}{3} = \frac{b}{3} (Y^3 - y^3) \dots\dots\dots (3).$$

The most convenient and systematic method of finding these values is by arranging the computation in tabular form. A few examples of each tabular method are taken for illustration.

CASE 7. MOMENT OF INERTIA OF A TEE-IRON,  $7'' \times 6'' \times 1''$ , ABOUT NEUTRAL AXIS  $XX$ .

The section is shown in Fig. 156. To get the ordinate of centre of

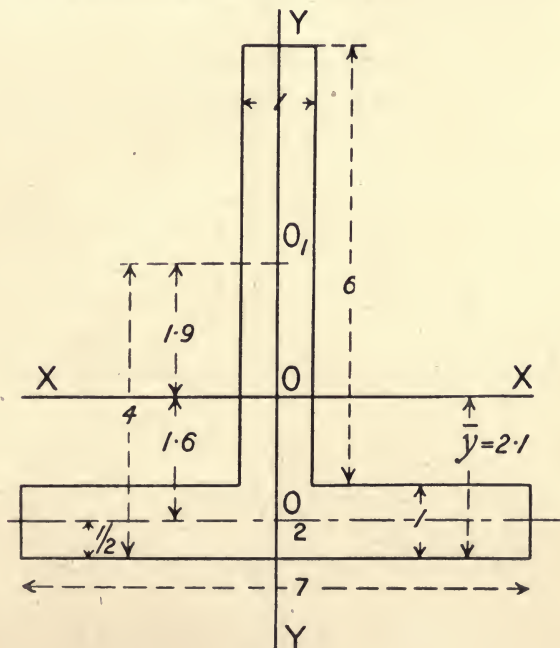


Fig. 156.

area, take moments about the lower edge of flange as axis. In this example, equation (1) is used to find moment of inertia

$$I_{xx} = A \left( \frac{d^2}{12} + y_0^2 \right).$$

$a$	$y$	$ay$	$y_0$	$y_0^2$	$\frac{d^2}{12}$	$a \left( y_0^2 + \frac{d^2}{12} \right)$
6	4	24	1.9	3.61	3	39.66
7	$\frac{1}{2}$	$3\frac{1}{2}$	1.6	2.56	$\frac{1}{12}$	18.48
13		$27\frac{1}{2}$				58.14
$\bar{y} = \frac{27\frac{1}{2}}{13} = 2.1''$			$I_{xx} = 58.14$			

CASE 8. MOMENT OF INERTIA OF A FLANGED SECTION (FIG. 157) ABOUT NEUTRAL AXIS  $XX$ .

This example will, for comparison, be worked out by both tabular methods.

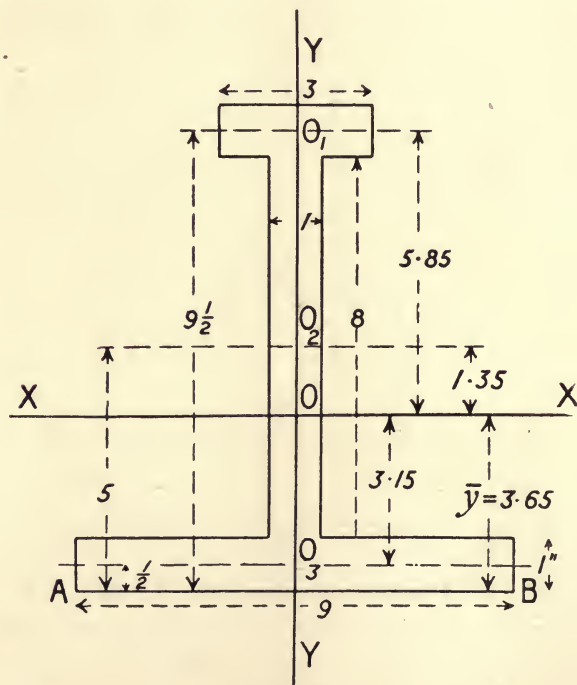


Fig. 157.

$O$  is the centre of area of the whole section,  $O_1, O_2, O_3$  the centres of area of top flange, web, and lower flange respectively.

$$1^\circ. \quad I_{xx} = A \left( y_0^2 + \frac{d^2}{12} \right).$$

Find the position of neutral axis by taking moments of area about the lower edge of bottom flange as axis.

Member	$a$	$y$	$ay$	$y_0$	$y_0^2$	$\frac{d^2}{12}$	$a \left( y_0^2 + \frac{d^2}{12} \right)$
Top Flange	3	$9\frac{1}{2}$	$28\frac{1}{2}$	5.85	34.23	$\frac{1}{12}$	102.9
Web	8	5	40	1.35	1.82	$\frac{16}{3}$	57.20
Lower Flange	9	$\frac{1}{2}$	$4\frac{1}{2}$	3.15	9.92	$\frac{1}{12}$	90.10
	20		73				250.2
$\bar{y} = \frac{73}{20} = 3.65''$				$I_{xx} = 250.2$			

2°. In the second tabular method,  $Y$  represents the larger of two successive values of  $y$ . The moment of area and moment of inertia are first calculated round  $AB$ , the lower edge of bottom flange, then

$$I_{xx} = I_{AB} - A\bar{y}^2.$$

The ordinates of the three rectangles from  $AB$  are 0, 1; 1, 9; 9, 10, and their breadths are 9, 1, 3.

$b$	$Y$	$Y-y$	Area	$Y^2$	$Y^2-y^2$	$\frac{b}{2}(Y^2-y^2)$	$Y^3$	$Y^3-y^3$	$\frac{b}{3}(Y^3-y^3)$
9	0	1	9	0	1	4.5	0	1	3.0
1	1	8	8	1	80	40.0	1	728	242.6
3	9	1	3	81	19	28.5	729	271	271.0
	10			100			1000		
$A = 20$				$M_{AB} =$		73	$I_{AB} =$		516.6

$$\bar{y} = \frac{73}{20} = 3.65''$$

$$I_{xx} = I_{AB} - 20 \times 3.65^2 = 516.6 - 266.4 = 250.2$$

CASE 9. MOMENT OF INERTIA OF A GIRDER BUILT UP OF ANGLE-IRONS AND PLATES.

Fig. 158 gives the actual cross section of girder, and Fig. 159 shows the equivalent cross section in which the elements of area are reduced to a series of rectangles. Choose any line as axis, say, the lower edge of the bottom flange, the diagram and table show the breadths and ordinates measured up from this line  $AB$ , then using equations (2) and (3) we can get the moment of area  $M_{AB}$ , and moment of inertia  $I_{AB}$  about axis  $AB$ , and the distance of neutral axis  $\bar{y} = \frac{M_{AB}}{A}$ . Also the moment of inertia about neutral axis  $XX$

$$I_{XX} = I_{AB} - A\bar{y}^2 = I_{AB} - M_{AB}\bar{y}.$$

In table,  $Y$  represents the larger of two successive values of  $\bar{y}$ .

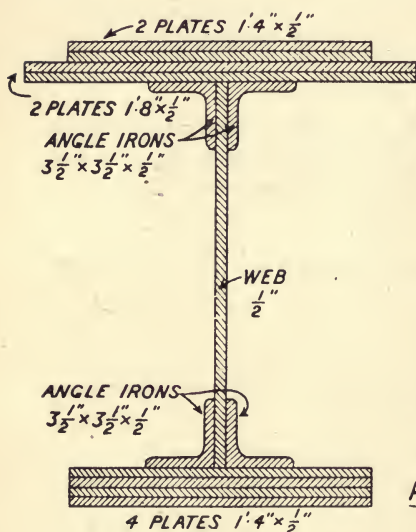


Fig. 158.

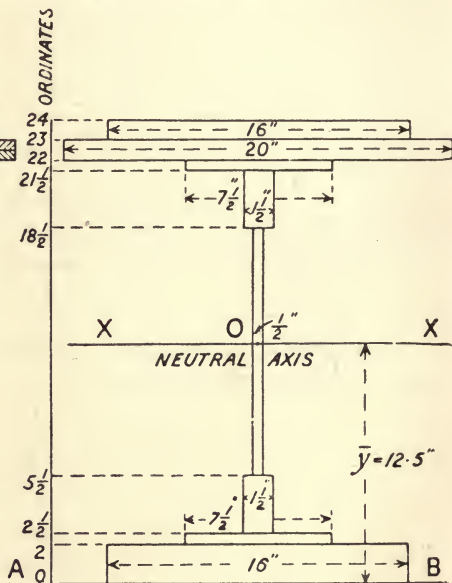


Fig. 159.

The axis  $AB$  round which the moment of inertia is first calculated has in this example been taken as the lower edge of bottom flange, but in sections of this class there is an axis, usually through the centre of depth, which is an axis of symmetry for several of the rectangles taken together, although not for all. In such cases arithmetical work will be saved (owing to symmetry) by first computing the moment of area and moment of inertia of the whole surface round this axis; as the ordinates measured above this axis are positive, and those below negative, the sum of the moments of area of the symmetrical elements is zero.

$b$	$Y$	$Y-y$	Area	$Y^2$	$Y^2-y^2$	$\frac{b}{2}(Y^2-y^2)$	$Y^3$	$Y^3-y^3$	$\frac{b}{3}(Y^3-y^3)$	
	0			0			0			
16	2	2	32	4	4	32	8	8	42.7	
$7\frac{1}{2}$	$2\frac{1}{2}$	$\frac{1}{2}$	$3\frac{3}{4}$	6.25	2.25	8.5	15.6	7.6	19.0	
$1\frac{1}{2}$	$5\frac{1}{2}$	3	$4\frac{1}{2}$	30.25	24	18	166.4	150.8	75.4	
$\frac{1}{2}$	18 $\frac{1}{2}$	13	$6\frac{1}{2}$	342.25	312	78	6331.6	6165.2	1027.5	
$1\frac{1}{2}$	$21\frac{1}{2}$	3	$4\frac{1}{2}$	462.25	120	90	9938.4	3606.8	1803.4	
$7\frac{1}{2}$	22	$\frac{1}{2}$	$3\frac{3}{4}$	484	21.75	81.5	10648	709.6	1774.0	
20	23	1	20	529	45	450	12167	1519	10127.0	
16	24	1	16	576	47	376	13824	1657	8837.3	
$A =$			91	$M_{AB} =$			1134	$I_{AB} =$		23706

$$\bar{y} = \frac{1134}{91} = 12.5''$$

$$\begin{aligned}
 I_{xx} &= I_{AB} - A\bar{y}^2 = 23706 - 91 \times 12.5^2 \\
 &= 23706 - 14218 \\
 &= 9488
 \end{aligned}$$

#### CASE 10. MOMENTS OF INERTIA OF THE SECTION OF AN UNEQUAL SIDED ANGLE-IRON.

Let the angle-iron be  $7'' \times 3\frac{1}{2}'' \times \frac{1}{2}''$  as in Fig. 160.

To find the moments of inertia about the axes  $XOX$  and  $YOY$ , through  $O$  the centre of area.

To find  $XOX$ , moments of area are taken about  $AB$ , to find  $YOY$ , they are taken round  $AC$ .

$a$	$y$	$ay$	$y_0$	$y_0^2$	$\frac{d^2}{12}$	$a\left(y_0^2 + \frac{d^2}{12}\right)$
3.5	3.5	12.2	1	1	4.1	17.85
1.5	.25	.3	2.25	5.06	.02	7.62
5.0		12.5				25.47
$\bar{y} = \frac{12.5}{5} = 2.5''$			$I_{xx} = 25.47$ inch units			

$a$	$x$	$ax$	$x_0$	$x_0^2$	$\frac{d^2}{12}$	$a \left( x_0^2 + \frac{d^2}{12} \right)$
3.5	.25	.88	.53	.28	.02	1.05
1.5	2	3.00	1.22	1.49	.75	3.36
5.0		3.88				4.41
$\bar{x} = \frac{3.88}{5} = 0.78''$			$I_{yy} = 4.41$ inch units			

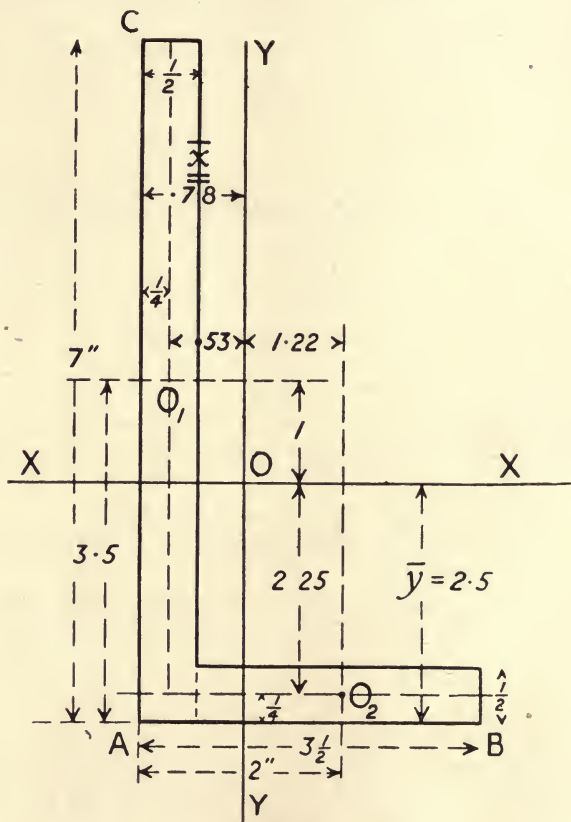


Fig. 160.

66. Graphic method of obtaining the moment of inertia of an irregular section.

In this method an area is easily constructed which is proportional to the moment of inertia of the section. To find the moment of inertia

about the axis  $XOX$ , through centre of area. For the upper portion of the section (Fig. 161) draw a line  $OY$  perpendicular to  $XOX$ , and at some whole number of units " $b$ " from  $O$ , draw  $LM$  parallel to  $XOX$ . Draw any number of lines in the area, of which  $PQ$  is one parallel to  $XOX$ . Make  $LN = OP$ , or  $PN = OL = b$ . Join  $NQ$  meeting  $LM$  in  $M$ . Join  $OM$  meeting  $PQ$  in  $R$ . If this is done for all the lines such as  $PQ$ , then  $R$  will trace out a curve the area of which is proportional to the moment of inertia.

If  $A_1$  is the total area of this new figure above and below  $XOX$ , then

$$I_{XX} = b^2 A_1.$$

*Proof.*

$$\frac{PQ}{LM} = \frac{PN}{NL},$$

and

$$\frac{LM}{PR} = \frac{OL}{OP};$$

multiplying, we get, since  $PN = OL = b$ ,

$$\frac{PQ}{PR} = \frac{b^2}{OP^2}.$$

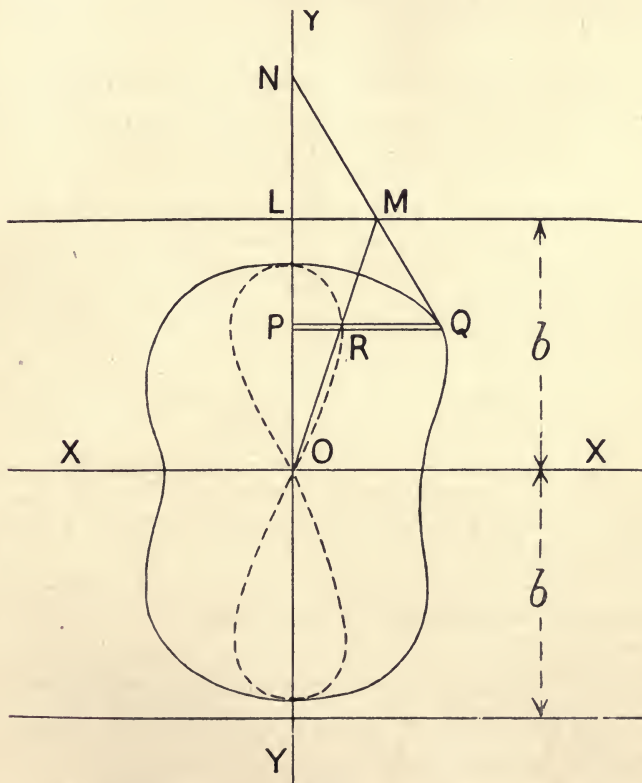


Fig. 161.

Therefore  $b^2 \cdot PR = PQ \cdot OP^2 \dots\dots\dots(1).$

Now if the coordinates of  $Q$  are  $x$  and  $y$ , and the coordinates of  $R$  are  $x_1$  and  $y$ , then

$$OP = y, \quad PQ = x, \quad PR = x_1,$$

and equation (1) becomes  $b^2 x_1 = xy^2 \dots\dots\dots(2).$

Considering an elementary strip of area at  $PQ$  of very small thickness  $dy$ , which is the same for both the  $Q$  and  $R$  curves; then moment of inertia of this strip about  $XOX$  is  $xdy \cdot y^2$  and the moment of inertia of whole area

$$I_{xx} = \int xy^2 dy = b^2 \int x_1 dy. \quad \text{From (2)} \\ = b^2 A_1,$$

or moment of inertia about  $XOX$  is the area traced out by  $R$ , multiplied by  $b^2$ .

### EXERCISES.

1. The dimensions of the section of a cast-iron beam are :—Top flange 4 inches by  $1\frac{1}{2}$  inches; bottom flange 12 inches by  $1\frac{3}{4}$  inches; web 16 inches by  $1\frac{1}{2}$  inches. Calculate the moment of inertia of the section about the neutral axis.

*Ans.* 2278·40.

2. Determine the moment of inertia and radius of gyration, about an axis through centre of area parallel to the flanges, of two tee-irons, each  $8'' \times 3'' \times 1''$ , placed flange to flange in cross shape.

*Ans.* 47·3.

3. The section of a plate girder is as follows :—Top flange, two plates each, 16 inches by  $\frac{1}{2}$  inch; bottom flange, three plates each, 14 inches by  $\frac{1}{2}$  inch; web plate 2 feet deep,  $\frac{3}{8}$  inch thick; angle-irons connecting flanges and web 4 inches by 4 inches by  $\frac{1}{2}$  inch. Find the moment of inertia of the section about neutral axis.

*Ans.* 8175.

4. Find the moment of inertia of the section of a circular tube 10 inches external diameter and 1 inch thick.

5. Show that in the case of a triangular area the moment of inertia about an axis through the centre of area is the same as if we consider one-third of the area concentrated at each of the middle points of the sides.

*Note.* This is useful in finding the moment of inertia of regular polygons which can be divided up into triangles.

6. Find the moment of inertia of a regular hexagon about an axis through its centre of area, and joining opposite angular points.

7. Find the moment of inertia of a rhomboidal or lozenge-shaped section about a diagonal as axis.

If  $b$  and  $h$  are diagonals,  $I = \frac{bh^3}{48}.$

## CHAPTER VII.

### GIRDERS.

#### 67. Modulus of section.

We have already proved that the relation between the bending moment and the stresses induced is given by the equation

$$M = \frac{f}{y} I,$$

i.e., the moment of the external forces is equal to the moment of internal stresses, or moment of resistance of the beam. The quantity  $\frac{I}{y}$  is called the "modulus of the section," and is generally denoted by the letter  $Z$ , and it is well to note that

$$Z = \frac{I}{y} = \frac{(\text{length units})^4}{\text{length}} = (\text{length units})^3.$$

Hence

$$M = \frac{f \cdot I}{y} = fZ.$$

Generally, it is necessary to consider the greatest tension and compression at any point of the cross section; then, if  $y_t$  and  $y_c$  are taken as the distances of the parts of the section furthest from the neutral axis, and if  $f_t, f_c$  be the tensile and compressive stresses corresponding to the distances  $y_t$  and  $y_c$ ,

$$\begin{aligned} M &= \frac{f_t}{y_t} I \text{ or } \frac{f_c}{y_c} I \\ &= f_t Z_t \text{ or } f_c Z_c. \end{aligned}$$

#### 68. Graphic method of finding the "equivalent area" and "modulus of section."

*Equivalent area.* On the section of a beam we have the stress varying from zero at the neutral axis to a maximum at the outer fibre. If now we replace the section of the beam by a section which has spread over it the same amount of stress, but instead of being variable, is constant all over and *equal to the stress on the outer fibre*, we get a figure which is called the EQUIVALENT AREA or MODULUS FIGURE.

Divide the beam section into a number of thin strips  $CEGK$  parallel to the neutral axis  $AB$  (Fig. 162).

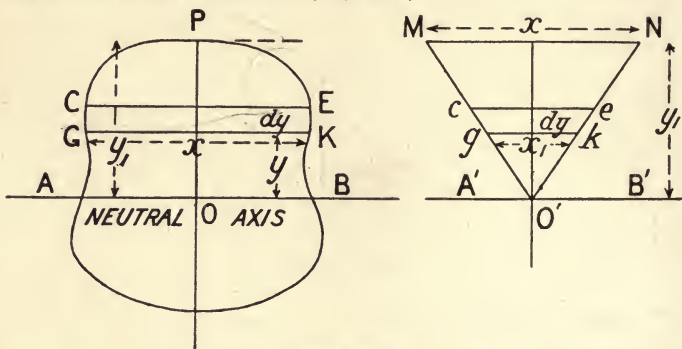


Fig. 162.

Let  $x = GK$ ,  $y$  and  $y_1$  the distances of  $GK$  and outer fibre at  $P$  respectively from the neutral axis, and let  $f$  and  $f_1$  be the stresses corresponding to  $y$  and  $y_1$ . Draw a line  $MN$  parallel to  $AB$  or its prolongation  $A'B'$ , and at a distance  $y_1$  from  $A'B'$ . Make  $MN = GK = x$ . Join  $O'M$ ,  $O'N$ , and let  $GK$ ,  $CE$  prolonged cut these lines in  $g$ ,  $k$ ,  $c$ ,  $e$ ; then the strip  $gkce$  has upon it a stress of constant intensity ( $f$ ), and the total stress on it is equal to the total stress on  $GKEC$ .

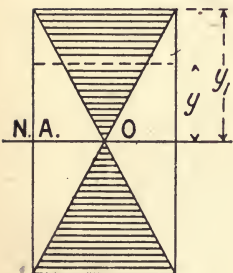


Fig. 163.

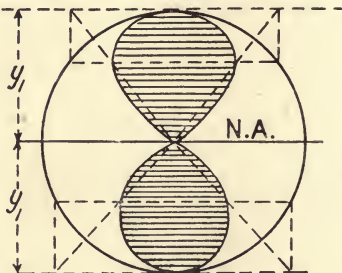


Fig. 164.

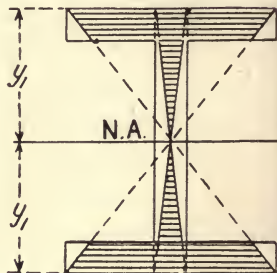


Fig. 165.

Let the thickness of the strip (same in both figures)  $= dy$ , and let  $gk = x_1$ .

By similar triangles,

$$\frac{gk}{MN} = \frac{y}{y_1} \text{ or } \frac{x_1}{x} = \frac{y}{y_1}, \text{ since } MN \text{ was made equal to } GK,$$

and

$$\frac{f}{f_1} = \frac{y}{y_1}.$$

Now total stress on  $GKEC = f x dy$

$$= \frac{f_1}{y_1} y x dy = \frac{f_1}{y_1} y_1 x_1 dy$$

$$= f_1 x_1 dy$$

$$= f_1 \times \text{area of } gkce = \text{total stress on } gkce.$$

Figs. 163, 164, 165 give the modulus figure for a rectangle, circle, and joist with equal flanges, which are all cases of *symmetrical* sections.

In the case of *unsymmetrical* sections, where the stress in the outer fibres is *not* the same in tension and compression, two figures can be drawn, one called the *tension figure* where the uniform stress is the maximum tensile stress; the other, the *compression figure* where the uniform stress is the maximum compressive stress. The neutral axis is not equidistant from the top and bottom, and the figures are drawn thus:—

*Compression figure* (Fig. 166). Draw a line on the lower side, at a distance from the neutral axis, equal to the distance of the upper fibre from the same axis. Complete the figure as before by projecting the widths on the lower line. Total stress =  $f_c \times \text{area of figure}$ .

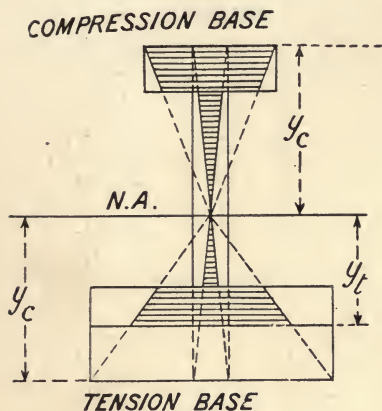


Fig. 166.

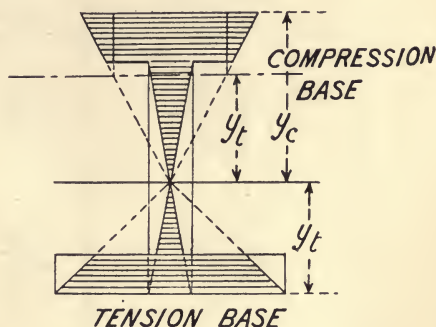


Fig. 167.

*For the tension figure* (Fig. 167). Draw a line parallel to the neutral axis, at a distance *above* it equal to the distance of lower fibre from that axis; project all the widths on to that line, and join up with centre of area. The total stress =  $f_t \times \text{area of figure}$ .

Fig. 168 gives the modulus figure for a tee-iron.

As the total amount of tension is equal to the total amount of compression, the area of the figure above the neutral axis must be equal to the area of the figure below the neutral axis, whether the section be symmetrical or not.

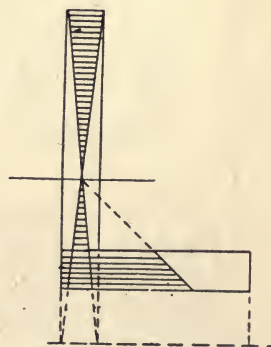


Fig. 168.

*Moment of Resistance.* Since the intensity of stress all over the figure is the same, the position of the resultant will be at the centre of gravity.

In Fig. 169 let  $a$  be the area of figure above neutral axis, which, as shown, must be equal to the area below the neutral axis.

$d$  = distance between the centres of gravity

$G_1$  and  $G_2$  of the upper and lower figures.

$f$  = intensity of stress.

Now, taking moments about neutral axis,

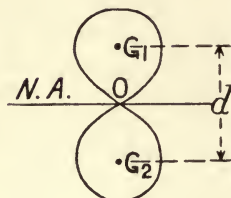


Fig. 169.

SECTION	MOMENT OF INERTIA $I$	MODULUS OF SECTION $Z$
	$\frac{bh^3}{12}$	$\frac{bh^2}{6}$
	$\frac{b^4}{12}$	$0.118 b^3$
	$\frac{\pi D^4}{64}$	$\frac{\pi D^3}{32}$ $= 0.0982 D^3$
	$\frac{\pi}{64} (D^4 - d^4)$	$\frac{\pi (D^4 - d^4)}{32 D}$ $= 0.0982 \frac{(D^4 - d^4)}{D}$
 HOLLOW RECTANGLE OR I WITH EQUAL FLANGES	$\frac{bh^3 - b_1h_1^3}{12}$	$\frac{bh^3 - b_1h_1^3}{6h}$

Moment of stress =  $f$  [area of top half figure  $\times OG_1$  + area of bottom half of figure  $\times OG_2$ ];

$$= f \times \text{area of either half} \times G_1 G_2;$$

$$= fad = fZ, \text{ where } Z = ad;$$

or moment of stress = stress on material  $\times$  modulus of the section.

The table on p. 156 gives the value of  $Z$  for some of the simpler forms of section.

### Examples.

1. A floor joist of  $I$  section with equal flanges, span 15 feet, is supported at both ends. The outside depth is 6 inches, breadth of flanges 4 inches, thickness of web and flanges each  $\frac{1}{2}$  inch. Find the uniformly-distributed load per foot run which the joist will carry, if stress is not to exceed 5 tons per square inch.

We have  $b = 4$  inches;  $b_1 = 3\frac{1}{2}$  inches;

$h = 6$  inches;  $h_1 = 5$  inches;

$$Z = \frac{I}{y_1} = \frac{4 \times 216 - 3\frac{1}{2} \times 125}{36} = 11.8 \text{ inches}^3.$$

Moment of stress =  $11.8 \times 5 = 59$  inch-tons.

Equating to this the value of the maximum bending moment  $\frac{wl^2}{8}$ , where  $w$  is in tons per foot run, we get

$$\frac{w 15^2 \times 12^2}{12 \times 8} = 59.$$

Therefore  $w = \frac{59 \times 2}{675} = 0.17$  tons per foot run.

2. A steel tube, 36 inches long, is supported at the end. External diameter, 2 inches; thickness,  $\frac{1}{4}$  inch. Find the central load which will produce a maximum stress of 8 tons per square inch.

$$Z = \frac{I}{y_1} = \frac{.0982 (2^4 - 1.5^4)}{2} = \frac{.0982 (16 - 5.06)}{2} = 0.537.$$

Therefore, maximum bending moment

$$\frac{W \times 36}{4} = 0.537 \times 8 = 4.3 \text{ inch-tons};$$

and

$$W = \frac{4.3}{9} = .48 \text{ tons.}$$

### 69. Beams of uniform strength.

In most cases the bending moment varies from section to section of a beam, and the sections are accordingly made deeper or broader where the greater bending moments come, being designed so that the maximum stress  $f$  allowed is uniform throughout the whole length of the beam;

when the section is so proportioned the beam is said to be of *uniform strength*. Now  $M=fZ$ , and as  $f$  is constant,  $Z$ , the strength modulus of section, must be proportional to  $M$ . Take the simplest case of a rectangular section :

1. Beam fixed at one end, loaded at the other with weight  $W$ .

$M=Wx$ , and  $bd^2 \propto x$ . If  $b$  is constant, then the elevation of beam showing  $d$  is a parabola. If  $d$  is constant, the plan showing  $b$  is a triangle.

2. Beam fixed at one end, loaded uniformly.

$M=\frac{w}{2}x^2$ , and  $bd^2 \propto x^2$ . If  $b$  is constant, the elevation showing  $d$  is a triangle. If  $d$  is constant, the plan showing  $b$  is a parabola.

3. Beam supported at the ends, loaded uniformly.

$M=\frac{w}{8}(l^2-4x^2)$ , and  $bd^2 \propto (l^2-4x^2)$ . If  $b$  is constant, the elevation showing  $d$  is an ellipse. If  $d$  is constant, the plan showing  $b$  is two parabolas.

## 70. Unsymmetrical sections. Beam with flanges and web.

If a section is symmetrical, the neutral axis passes through the centre of depth ; hence the maximum tensile stress on the material is *equal* to the maximum compressive stress. But some materials, such as cast iron, are five times as strong in compression as in tension ; consequently the area of the tension flange is made five times the area of the compression flange, and the neutral axis will be about five times as far from the compression flange as from the tension flange.

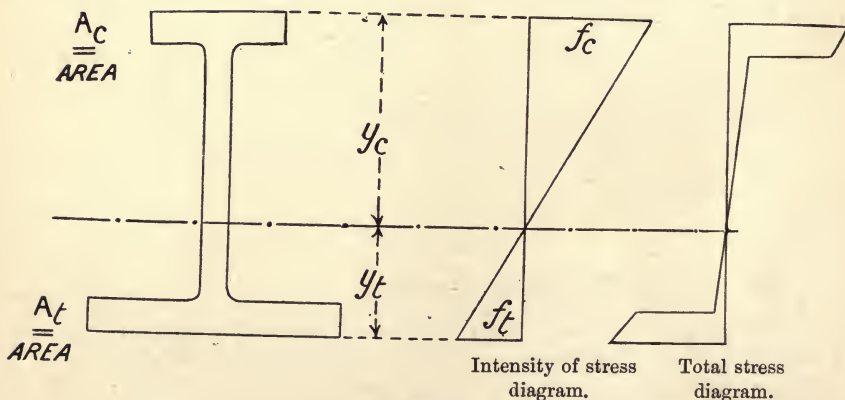


Fig. 170.

In Fig. 170, let

$A_c$  = area of compression flange,

$A_t$  = area of tension flange,

$f_c$  = maximum intensity of compressive stress,

$f_t$  = maximum intensity of tensile stress,

$y_c$  and  $y_t$  the distances from the neutral axis of the outer fibre in the compression and tension flanges respectively.

Then 
$$M = \frac{f_t}{y_t} I \text{ or } \frac{f_c}{y_c} I,$$

and, as  $\frac{f_t}{y_t} = \frac{f_c}{y_c}$ , we get the same value of  $M$  whichever we take.

Fig. 170 gives the general form of the *intensity of stress* diagram, and *total stress* diagram for an  $I$  section.

In cast-iron girders the web varies from 1 inch to 2 inches in thickness, and its moment of inertia must be taken account of in calculating the bending strength.

#### STEEL GIRDERS.

In girders of this class the two flanges are held apart by a thin, deep web as in the plate girder type, or by diagonal bracing as in lattice girders. *The function of the web or diagonal braces is to resist the shearing forces.* The *bending moment* is borne mainly by the *flanges*, one being in tension and the other in compression, and as the depth of each flange is small in comparison to the depth of the girder, the intensity of stress is practically uniform over the whole of each flange. In actual practice the flanges alone are taken as resisting the bending moment, the section of the web being neglected in the moment of resistance of the cross section.

#### Web. Plate Girder.

Let  $F$  = shearing force at any section in tons,

$d$  = depth of web in inches,

$t$  = thickness of web in inches,

$f_s$  = working shear stress in tons per square inch ;

then 
$$f_s = \frac{F}{\bar{d}t},$$

or 
$$t = \frac{F}{f_s \bar{d}}.$$

In practice, the thickness should not be less than  $\frac{3}{8}$  inch.

Angle-iron or tee-iron stiffeners are usually riveted to the web at intervals approximately equal to the depth of the girder. We have seen that shear stress on a square element is equivalent to a tensile stress along one diagonal and a compressive stress along the other, each at  $45^\circ$  with the direction of shear stress. The stiffeners are introduced to prevent buckling due to the compressive stress.

*Flanges.* Let  $A_c$ ,  $A_t$  be the areas of the flanges,  $f_c$ ,  $f_t$  the intensities of stress on them, and  $d$  the depth taken from the middle of one flange area to the middle of the other (Fig. 171).

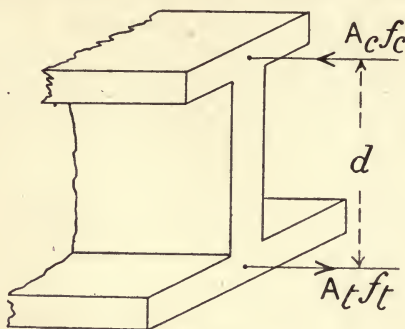


Fig. 171.

The total stress on each flange must be the same, or

$$A_c f_c = A_t f_t.$$

The forces  $A_c f_c$  and  $A_t f_t$ , being equal in value and opposite in sign, are the forces of the stress couple whose lever arm is  $d$ .

Therefore

$$M = A_c f_c d = A_t f_t d,$$

or, calling  $H$  the total horizontal stress on either flange, the bending moment

$$M = Hd,$$

or

$$H = \frac{M}{d}.$$

If, therefore, the depth of the girder is uniform, *i.e.*  $d$  constant, then  $H$  varies as  $M$ .

If the horizontal stress is uniform, *i.e.*  $H$  constant, then  $d$  varies as  $M$ .

We shall consider these two cases in detail, taking only the moments due to uniform *dead* load, for which the bending moment diagram is a parabola.

CASE 1. GIRDER WITH PARALLEL FLANGES IN WHICH THE DEPTH IS CONSTANT.

Here  $d$  is constant. Therefore  $H \propto M$ , and the *stress* diagram (Fig. 172) is a parabolic curve similar to the diagram of moments.

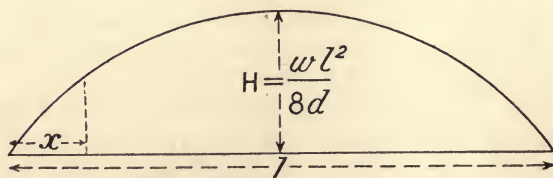


Fig. 172. Stress diagram.

The maximum flange stress at centre  $H = \frac{wl^2}{8d}$ ; at any other point of the span  $H = \frac{w}{2d}x(l-x)$ ; compressive in the top flange and tensile in the bottom flange.

If  $f$  = working intensity of stress,

$b$  = breadth of flange,

$t$  = thickness of flange,

then

$$H = A f = b \times t \times f.$$

Therefore

$$b \times t \times f \propto M.$$

As it is usual to make  $b$  constant, we get  $t \propto M$ , or, the thickness of the flange at any cross section varies as the bending moment at that section.

Hence, the flange is generally constructed by drawing the curve of bending moment to a given scale. Then, by altering the scale, the ordinates of the same curve represent the flange thicknesses.

### Example.

Girder 60 feet long, 5 feet deep, 3 feet wide, carries a uniform load of 3 tons per foot run. Find the thickness of plates required for the flanges, taking 5 tons per square inch as the maximum intensity of stress.

$$\begin{aligned} \text{Max. } M &= \frac{wl^2}{8} = \frac{3 \times 3600}{8} \text{ foot-tons} \\ &= 1350 \text{ foot-tons.} \end{aligned}$$

Then

$$\begin{aligned} 1350 &= f b d t \\ &= 5 \times 144 \times 3 \times 5 \times t, \\ \therefore t &= .125 \text{ feet} = 1\frac{1}{2} \text{ inches,} \end{aligned}$$

or thickness at the centre =  $1\frac{1}{2}$  inches.

Suppose, as in Fig. 173, this thickness is made up of three  $\frac{1}{2}$ -inch plates. Divide the central ordinate into three equal parts, and draw horizontal lines; the points where these cut the moment curve give

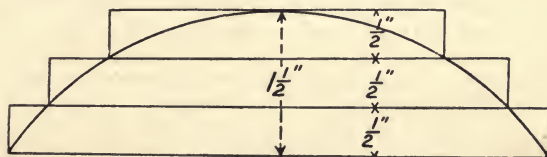


Fig. 173.

the required *lengths* of the plates. The stepped outline represents the moment of resistance, which for *each* plate

$$= f b t d = 5 \times 36 \times \frac{1}{2} \times 5 \times 12 = 5400 \text{ inch-tons} = 450 \text{ foot-tons,}$$

and as there are three plates the total moment of resistance

$$= 3 \times 450 = 1350 \text{ foot-tons,}$$

which is equal to the maximum bending moment.

## CASE 2. PARABOLIC GIRDERS.

If  $H$  be constant, then  $d \propto M$ .

The depth of the girder is everywhere proportional to the bending moment, and the *shape of girder* is that of the moment diagram, a parabola (Fig. 174).

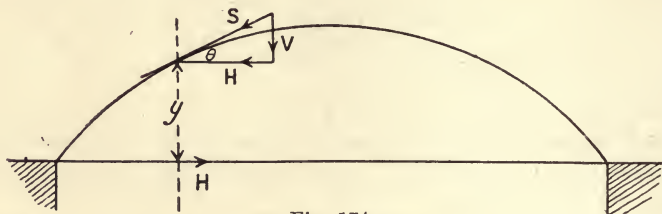


Fig. 174.

The uniform horizontal stress in the lower flange, or the *horizontal component* of the inclined stress at *any* point in the curved flange, is

$$H = \frac{wl^2}{8d}.$$

The *vertical component*

$$V = H \tan \theta.$$

The total stress at any point in the curved flange is

$$S = H \sec \theta.$$

Hence total stress  $S$  varies as  $\sec \theta$ ; consequently greater thickness is required at the ends than at the centre, for  $\sec \theta$  increases as  $\theta$  increases.

$$\begin{aligned} \text{At the top where } \theta = 0, \quad S &= H; \\ \text{at } \theta = 45^\circ, \quad S &= \sqrt{2} H. \end{aligned}$$

## FUNCTION OF WEB IN RESISTING THE SHEARING FORCE.

## Case I.

In parallel girders (as in Fig. 175) the stress on the flanges is *horizontal*; consequently the *vertical* shearing force can only be resisted

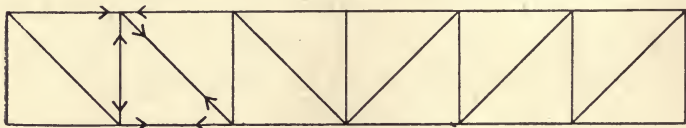


Fig. 175.

by the vertical component of the stresses in the diagonal bars or web, and it is the horizontal components of the same stresses which produce the increment of flange stress at each panel point. The web or inclined bracing is therefore absolutely necessary in parallel girders.

*Case II.*

In the case of parabolic girders, the horizontal component of the stress at any point of the curved flange, and the stress on the horizontal flange, are constant.

Referring to Figs. 174 and 176, the bending moment at any point whose coordinates are  $x, y$ , is

$$M = Hy,$$

and the shearing force

$$F = \frac{dM}{dx} = H \frac{dy}{dx} = H \tan \theta;$$

but the vertical component of stress in curved flange  $= H \tan \theta$ .

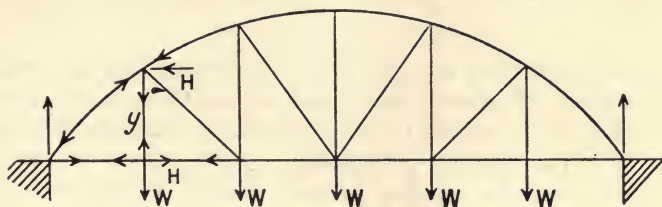


Fig. 176.

Therefore this vertical component suffices to balance the shearing force on the section. *No stress therefore occurs in the diagonal web bracing when the girder is uniformly loaded with a dead load.* Hence in such a girder *subject to a uniform dead load* the diagonal bracing may be omitted.

In the horizontal flange the stress is entirely due to the thrust of the ends of the curved flange, and might be replaced by abutments. This is the principle of *the arch* (Fig. 177).

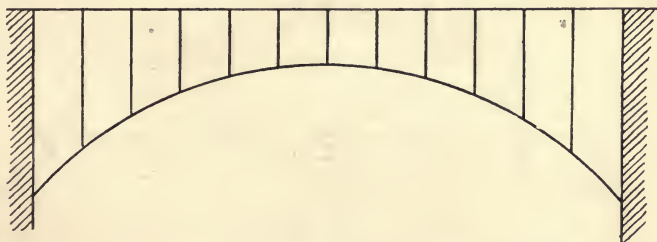


Fig. 177.



Fig. 178.

If the bridge have the curved flange *below* the horizontal the same reasoning applies, except that the stresses are reversed, the horizontal flange being now in compression, and the curved flange in tension (Fig. 178).

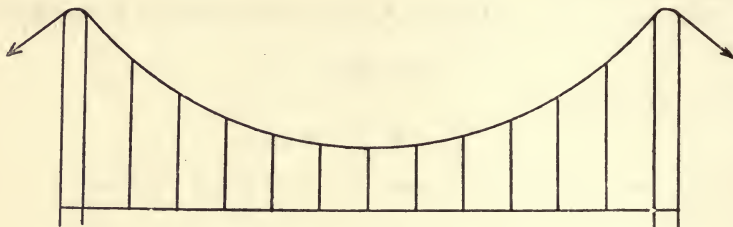


Fig. 179.

The horizontal flange can therefore be replaced by anchorages at the abutments. This is the principle of the *suspension bridge* (Fig. 179).

When the load, instead of being distributed throughout the entire length, is carried on cross-girders at the panel points, the same principles hold good; the bending moment diagram in this case being a polygon inscribed in a parabola.

### 71. Graphic method of finding the maximum stress in the members of a parabolic girder for a live load.

Let Fig. 180 represent a girder of this class, the intersections of the members of the upper flange lying on a parabola.

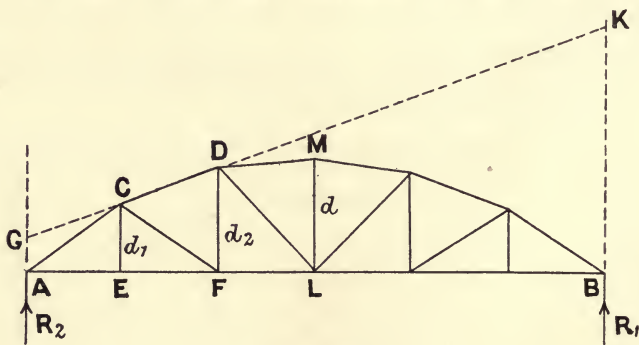


Fig. 180.

Let  $w$  be the intensity of the load per foot run.

Let  $AB$  be the span of length  $l$ , divided into say 6 equal bays.

$R_1$  and  $R_2$  the reactions at  $B$  and  $A$  respectively.

$d$  the central ordinate  $LM$ .

$d_1$  and  $d_2$  the ordinates  $CE$  and  $DF$  respectively.

$b$  the length of a bay  $= \frac{l}{6}$ .

$H$  the horizontal component of stress in upper flange.

*Flanges.* The maximum stress in flanges occurs when the whole span is loaded. Further, we know that in the case of a parabolic girder loaded with a uniform load, the horizontal component of stress in the flanges is constant.

To find the maximum stress in any member of the upper flange, say  $CD$ .

Produce the chord member  $CD$  to meet the verticals through  $A$  and  $B$  in  $G$  and  $K$  respectively; then

$$GK = l \sec \theta \dots\dots\dots(1),$$

where  $\theta$  is the inclination of  $CD$  to the horizontal.

But we have previously shown that the

$$\text{Stress in } CD = H \sec \theta \dots\dots\dots(2).$$

Therefore, from (1) and (2) we see, that the intercept  $GK$  represents the total stress in  $CD$  on the same scale that  $l$  represents the horizontal component

$$H = \frac{wl^2}{8}.$$

To find the maximum stress in braces. (Live load.)

Taking a bay as unit length, let  $W = wb$  be the load carried at the joints of the horizontal chord.

The maximum stress in brace  $CF$  (Fig. 180) occurs when all the joints to the right of  $F$  are loaded, then

$$R_2 = \frac{10}{6} W.$$

The horizontal component of stress in  $CF$

$$= \frac{\text{moment at } D}{d_2} - \frac{\text{moment at } C}{d_1};$$

$$\text{moment at } D = \frac{20}{6} Wb; \quad \text{moment at } C = \frac{10}{6} Wb.$$

Now the ordinates being proportional to the bending moments,

$$\frac{d_1}{d} = \frac{CE}{ML} = \frac{\frac{wb \times 5b}{2}}{\frac{w \times 6b \times 6b}{8}} = \frac{5}{9},$$

or,

$$d_1 = \frac{5}{9} d; \quad \text{and} \quad d_2 = \frac{8}{9} d.$$

Therefore, maximum horizontal component of stress in  $CF$

$$= \frac{\frac{20 Wb}{6}}{\frac{8}{9} d} - \frac{\frac{10 Wb}{6}}{\frac{5}{9} d} = \frac{3}{4} \frac{Wb}{d} \dots\dots\dots(3).$$

But the horizontal component of stress in flanges when the girder is uniformly loaded is

$$H = \frac{wl^2}{8} = \frac{w \times 6b \times 6b}{8} = \frac{18}{4} \frac{wb^2}{d} = \frac{18}{4} \frac{Wb}{d} \dots\dots\dots(4).$$

Therefore, from (3) and (4) we see that the maximum horizontal component of stress in inclined brace

$$= \frac{H}{6}.$$

Now the horizontal projection of any brace  $= b = \frac{l}{6}$ .

Therefore, *the actual length of each brace represents the maximum stress in it, on the same scale that  $l$  represents the constant horizontal component of stress ( $H$ ), the girder being uniformly loaded.*

**72. Application of method of sections to bridge trusses.** (Dead and live loads.)

*Stresses in web and chord members.* In applying the method of sections to a girder such as is shown in Fig. 181, a vertical or nearly vertical section cuts one bar only of the web, and since each of the top and bottom chords is horizontal the stress upon the web bar must be such that its vertical component is equal to the shearing force  $F$  at the section; so that the stress in inclined web member is

$$F \operatorname{cosec} \theta,$$

$\theta$  being the angle which the member makes with horizontal.

Again, the stress in any top or bottom member is proportional to the bending moment  $M$  at a vertical section taken through the opposite joint. If  $d$  = depth of the girder, then the stress

$$S = \frac{M}{d}.$$

*Counter bracing.* The diagonals in the Pratt truss (Fig. 181) are all ties or tension members; but in certain positions of the live load, those near the centre of span would be subjected to compression. In order to prevent this, another diagonal sloping in the opposite direction (shown dotted in Fig. 181) is inserted in those panels where this reversal of stress takes place. These panels in which two diagonals occur are said to be *counter braced*, and the additional diagonal is called a *counter brace*. Both these diagonals are tension members; one only being in action at a time, and the minimum stresses for these diagonals in a counter braced panel are zero.

*Example.*

A Pratt truss, 112 ft. span (Fig. 181), divided into 8 bays, carries a dead load of  $\frac{3}{4}$  ton per foot run, and a live load of  $1\frac{1}{4}$  tons

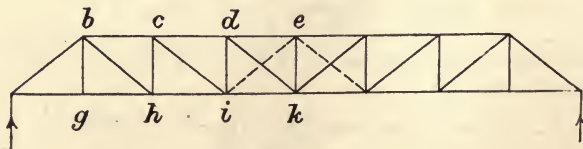


Fig. 181.

per foot run, both supported at the joints of the bottom chord. The depth of the girder is  $\frac{1}{10}$ th of the span. Determine the stresses in the chord members  $de$  and  $ik$ , the diagonal tie  $ci$ , and the verticals  $bg$ ,  $ch$ .

Length of each bay =  $\frac{11 \cdot 2}{8} = 14$  feet.

Depth of girder =  $\frac{11 \cdot 2}{10} = 11 \cdot 2$  feet.

Dead load at each bottom joint = 10·5 tons.

Live load at each bottom joint = 17·5 tons.

If  $\theta$  = angle of inclination of each tie with horizontal, then

$$\tan \theta = 1 \cdot 25, \quad \operatorname{cosec} \theta = 1 \cdot 6.$$

*Stress in members  $de$  and  $ik$ .* The maximum stress in each member of the chords occurs when the span is fully loaded with both the dead and live loads.

The reactions at each support =  $\frac{1}{2} \cdot 7 \cdot 28 = 98$  tons.

*Member  $ik$ , bottom chord.* Let  $S$  = stress.

Take moments about joint  $d$ , upper chord,

$$\begin{aligned} S \times 11 \cdot 2 = M &= 98 \times 42 - 28(2 + 1) 14 \\ &= 2940 \text{ foot-tons.} \end{aligned}$$

$$\text{Therefore } S = \frac{2940}{11 \cdot 2} = 262 \cdot 5 \text{ tons tension.}$$

*Member  $de$ , top chord.* Let  $S'$  = stress.

Take moments about joint  $k$  of lower chord,

$$\begin{aligned} S' \times 11 \cdot 2 = M' &= 98 \times 4 \times 14 - 28(3 + 2 + 1) 14 \\ &= 3136 \text{ foot-tons.} \end{aligned}$$

$$\text{Therefore } S' = \frac{3136}{11 \cdot 2} = 280 \text{ tons compression.}$$

*Diagonal tie  $ci$ . Stress due to dead load.*

The reactions due to dead load are each equal to

$$\frac{1}{2} \times 7 \times 10 \cdot 5 = 36 \cdot 75 \text{ tons.}$$

Take a vertical section cutting  $cd$ ,  $ci$ ,  $hi$ .

The shearing force  $F_1$  at the section = 15·75 tons.

Let  $S_1$  be the stress in  $ci$  due to dead load, then

$$S_1 \sin \theta = F_1,$$

$$\text{or, } S_1 = F_1 \operatorname{cosec} \theta = 15 \cdot 75 \times 1 \cdot 6 = 25 \cdot 2 \text{ tons tensile.}$$

*Diagonal tie  $ci$ . Stress due to live load.*

The maximum stress in diagonal  $ci$  occurs when all the joints to the right of  $i$  are loaded, the joints  $g$  and  $h$  being unloaded.

The shearing force  $F_2$  at the vertical section cutting  $ci$  is then equal to the reaction at the left support.

$$\begin{aligned} \text{Therefore } F_2 &= \frac{17 \cdot 5}{8} \{1 + 2 + 3 + 4 + 5\} \\ &= \frac{17 \cdot 5 \times 15}{8} = 32 \cdot 8 \text{ tons.} \end{aligned}$$

Let  $S_2$  = maximum tensile stress.

$$\begin{aligned}\text{Then } S_2 &= F_2 \operatorname{cosec} \theta = 32.8 \times 1.6 \\ &= 52.48 \text{ tons tensile.}\end{aligned}$$

The *minimum stress* in *ci* occurs when the joints *g*, *h* are loaded with the live load, the other joints being unloaded.

The shearing force on the section cutting *ci* is then equal to the reaction at the right support.

$$\begin{aligned}\text{That is, } F_2' &= \frac{3}{8} 17.5 = 6.56 \text{ tons,} \\ \text{and } S_2' &= 6.56 \times 1.6 = 10.5 \text{ tons compressive.}\end{aligned}$$

*Stress due to both dead and live loads.*

The stresses due to both loads are got by adding the dead load stress to each of the live load stresses, thus :

Web member	<i>ci</i>	
Dead load	25.20	tensile
Live load greatest	52.48	tensile
Live load least	10.50	compressive
Maximum	77.68	tensile
Minimum	14.70	tensile

Both the maximum and minimum being of the same sign no counter brace will be required in this panel.

*Stress in vertical bg.*

The stress in *bg* is simply the weight at the joint *g*,  
 stress due to dead load = 10.5 tons *tension*,  
 stress due to live load = 17.5 tons *tension*.

*Stress in vertical ch.*

Take an approximately vertical section cutting members *bc*, *ch*, *hi*.

The stress in *ch* is the vertical component of stress of *ci*, and the compressive and tensile stresses due to live load are produced by the same distribution of the live loads as taken for *ci*.

Therefore,

$$\begin{aligned}\text{stress due to dead load} &= F_1 = 15.75 \text{ tons compression,} \\ \text{maximum compressive stress due to live load} &= F_2 = 32.8 \text{ tons,} \\ \text{maximum tensile stress due to live load} &= F_2' = 6.56 \text{ tons.}\end{aligned}$$

*Verticals.*

	<i>bg</i>		<i>ch</i>	
Dead load	10.5 tons	tension	15.75 tons	compression
Live load	0 "	compression	32.80 "	compression
Live load	17.5 "	tension	6.56 "	tension
Maximum	28.0 "	tension	48.55 "	compression
Minimum	10.5 "	tension	9.19 "	compression

**73. Effect of live load on a girder shown graphically.**

Let Fig. 182 represent a Pratt truss of 9 panels subject to uniform live load equivalent to 15 tons at each panel point, the central panel being counter braced. The stresses in the chord members are found graphically as for a dead load; the stresses in these members being a maximum when the bridge is fully loaded.

Fig. 183 shows the stress diagram for the half girder in this case.



Fig. 182.

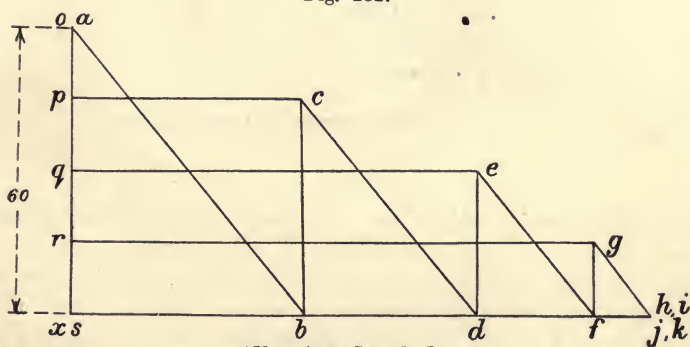


Fig. 183.

To show the effect of the live load on the stresses in braces and verticals; Fig. 184 gives the stress diagram when the *six* bottom joints to the *right* are loaded, the two joints on the left being unloaded, and we see that the braces, including the *full line counter brace IJ*, are in *tension* and the verticals in *compression*. There is no stress in the dotted counter brace.

Fig. 185 shows the stress diagram when the *two joints to the left* are loaded, the remaining six joints being unloaded. We see from this figure that the braces *AB* and *CD* are in tension, and the vertical

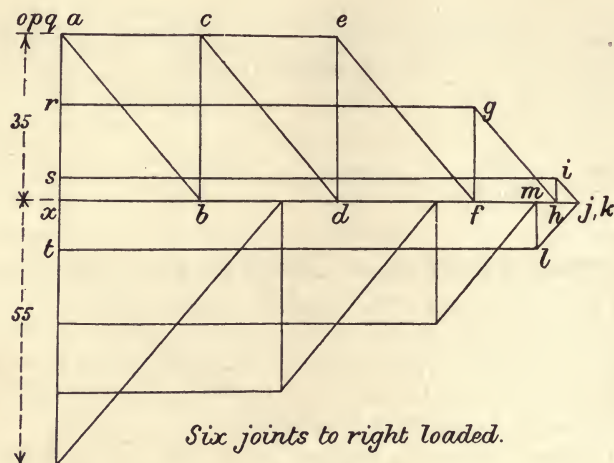


Fig. 184.

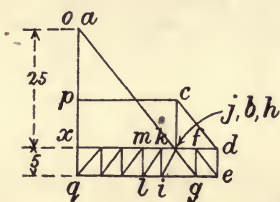


Fig. 185.

*BC* in compression as before, but the stresses in the braces *EF*, *GH* are now *compressive*, and the stresses in the verticals *DE*, *FG* are *tensile*; also that the dotted counter brace *IJ* is now in tension and there is no stress in the full line counter brace. The *maximum tensile stress* in brace *EF* (see Fig. 184) occurs when all the joints to the right of it only are loaded, and from Fig. 185 we see that when the two joints to the left of it only are loaded we get compressive stress in it.

#### *Summary.*

The maximum stress in any vertical or diagonal brace occurs when the live load covers the longer segment of the span.

The maximum stress in any counter brace occurs when the live load covers the shorter segment.

The minimum stress in a vertical or diagonal brace occurs when the live load covers the shorter segment of the span.

The minimum stress in a counter brace is zero.

### Examples.

1. A cast-iron girder of the section in Fig. 186, 20 feet span, is supported at both ends. If the maximum intensity of tensile stress is 1 ton per square inch, find what uniformly-distributed load the girder will carry.

The following data are given, but they should be checked by the student:

Distance of neutral axis from lower edge of bottom flange,  $\bar{y} = 3.13$  inches.

Moment of inertia of section  $I = 220.4$  inches<sup>4</sup>.

The maximum bending moment,

$$M = \frac{wl^2}{8} = I \frac{f_t}{y_t}$$

$$y_t = \frac{3.13}{12} \text{ feet}; f_t = 144 \text{ tons per square foot};$$

$$\therefore wl \frac{l}{8} = \frac{220.4}{144 \times 144} \times \frac{144}{\left(\frac{3.13}{12}\right)} \text{ foot-tons};$$

$$\therefore \text{total load} = wl = \frac{220.4}{93.9} = 2.35 \text{ tons.}$$

2. A flanged girder, 50 feet span, 4 feet deep, flanges 12 inches broad, carries a uniform load of  $1\frac{1}{2}$  tons per foot run; working stress 6 tons per square inch. Find the necessary thickness of flange at  $\frac{1}{8}$ ,  $\frac{1}{4}$ , and  $\frac{1}{2}$  the span.

$$M = Afd = btf d.$$

(a) At  $\frac{1}{8}$  of the span

$$M = \frac{w}{2} (l - x) x = btf d;$$

$$\therefore \frac{3}{4} \cdot \frac{7}{8} 50 \cdot \frac{5.0}{8} = 1 \times t \times 6 \times 144 \times 4 \text{ foot-tons};$$

$$t = 0.06 \text{ foot} = 0.7 \text{ inch.}$$

(b) At  $\frac{1}{4}$  of the span

$$\frac{w}{2} x (l - x) = btf d;$$

$$\therefore \frac{3}{4} \cdot \frac{5.0}{4} \cdot \frac{3}{4} 50 = 1 \times t \times 6 \times 144 \times 4 \text{ foot-tons};$$

$$t = 0.1 \text{ foot} = 1.2 \text{ inches.}$$

(c) At  $\frac{1}{2}$  the span

$$\frac{3}{4} \cdot \frac{5.0}{2} \cdot \frac{5.0}{2} = 1 \times t \times 6 \times 144 \times 4 \text{ foot-tons};$$

$$t = 0.137 \text{ feet} = 1.64 \text{ inches.}$$

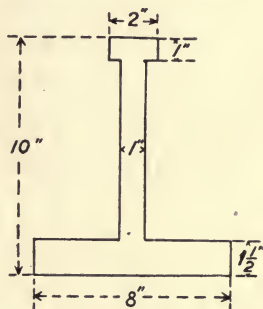


Fig. 186.

3. A girder, 40 feet span, 4 feet deep, carries a load of 12 tons at the centre; breadth of flanges 9 inches; working stress 6 tons per square inch. Find the thickness of the flanges at  $\frac{1}{4}$  and  $\frac{1}{2}$  the span.

(a) At  $\frac{1}{2}$  the span

Thickness  $\propto M$ ;

$$\therefore \text{Thickness at centre} \propto \frac{Wl}{4};$$

$$\therefore \frac{Wl}{4} = f b t d;$$

$$\therefore \frac{12 \times 40}{4} = 6 \times 144 \times \frac{3}{4} \times t \times 4;$$

$$t = \frac{10}{216} \text{ feet} = \frac{5}{8} \text{ inches (app.)}.$$

(b) At  $\frac{1}{4}$  the span

$$\frac{1}{2} \frac{Wl}{4} = 6 \times 144 \times \frac{3}{4} \times t \times 4;$$

$$\therefore t = \frac{5}{128} \text{ feet} = \frac{5}{16} \text{ inches (app.)}.$$

4. A steel girder, 40 feet span, 4 feet deep, carries a uniform load of 3 tons per foot run; width of flanges, 9 inches; working stress on upper flange = 6 tons per square inch; working stress on lower flange = 5 tons per square inch. Find the thickness of the flanges at the centre of span, and at 10 feet from each end.

Let  $t_c$  = thickness of compression flange,

$t_t$  = thickness of tension flange.

Then  $9t_c \cdot 6 = 9t_t \cdot 5$ .

$$\therefore \frac{t_c}{t_t} = \frac{5}{6}.$$

(a) At centre,

*Compression Flange:*

$$M = Hd = \frac{wl^2}{8};$$

$$\therefore \frac{3 \times 40 \times 40 \times 12 \times 12}{12 \times 8} = 6 \times 9 \times t_c \times 4 \times 12 \text{ inch-tons};$$

$$t_c = \frac{2.5}{9} \text{ inches} = \underline{2.8 \text{ inches.}}$$

*Tension Flange:*

$$\frac{3 \times 40 \times 40 \times 12}{8} = 5 \cdot 9 \cdot t_t \cdot 4 \times 12;$$

$$t_t = \underline{3\frac{1}{3} \text{ inches.}}$$

(b) At 10 feet from either end,

$$\frac{w}{2} x(l-x) = Hd;$$

$$\frac{3}{2} \cdot 10 \cdot 30 \cdot 12 = 6 \cdot t_c \cdot 9 \cdot 4 \cdot 12;$$

$$t_c = \frac{2.5}{12} = \underline{2.08 \text{ inches}};$$

$$t_t = \frac{6}{5} t_c = \underline{2.5 \text{ inches.}}$$

5. A steel plate girder, 60 feet span, 6 feet deep, carries a uniform load of  $\frac{3}{4}$  ton per foot run, and two loads of 8 tons each, placed 6 feet each side of the centre. Determine the necessary sections for top and bottom flanges, taking the working stress on top flange 6 tons per square inch, and on the bottom flange 5 tons per square inch. See Fig. 187.

The maximum bending moment is at the centre.

Bending moment at centre for uniform load =  $\frac{wl^2}{8} = 337.5$  foot-tons.

Bending moment at centre for concentrated loads =  $192.0$  foot-tons.

Total bending moment at centre =  $529.5$  foot-tons.

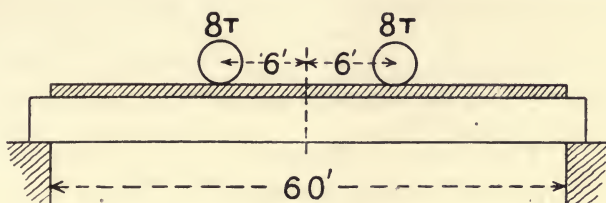


Fig. 187.

Flange stress at centre =  $\frac{529.5}{6} = 88.2$  tons.

The section is sketched in Fig. 188; the dimensions of flange will now be determined. It should be noted that only the areas of the parts of the angle-irons which are attached to the flanges are taken into consideration in finding the flange area; the parts of the angle-irons attached to web are omitted. Rivet holes are deducted in tension flange to get its net sectional area.

Area of top flange at centre of girder

$$= \frac{88.2}{6} = 14.7 \text{ square inches.}$$

Net section of bottom flange at centre of girder

$$= \frac{88.2}{5} = 17.7 \text{ square inches.}$$

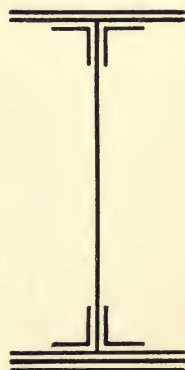


Fig. 188.

*Sections:*

Top flange	{	Angle-irons used, $4'' \times 4'' \times \frac{3}{8}''$ ,	
		giving area for flange	
		$2 \times 4'' \times \frac{3}{8}''$	= 3.0 square inches.
		2 plates $16'' \times \frac{3}{8}''$	= 12.0
		Total area	= <u>15.0</u>

$$\begin{array}{lcl}
 \text{Bottom flange} & \left\{ \begin{array}{l} \text{Angle-irons } 4'' \times 4'' \times \frac{3}{8}'', \\ \text{giving area for flange} \\ 2 \times 4'' \times \frac{3}{8}'' \\ 3 \text{ plates, } 16'' \times \frac{3}{8}'' \end{array} \right. & \begin{array}{l} = 3.0 \text{ square inches.} \\ \\ = 18.0 \\ \hline 21.0 \end{array} \\
 & & \text{Deduct } 8, \frac{3}{4}'' \text{ rivet holes} \\
 & & \text{in } \frac{3}{8}'' \text{ plates and angles} = 2.5 \\
 & & \text{Total net area} = \underline{18.5}
 \end{array}$$

These areas allow a slight margin in excess of the area required.

6. In a steel girder bridge of 150 feet span, with lattice bracing, the top and bottom flanges are of the box-type section. The stresses at two sections of the upper flange have been calculated to be 282 tons and 426 tons respectively. Design suitable cross sections at these points for a working stress of 6 tons per square inch. Take the top plate 3 feet wide and the side plates 2 feet deep.

At Section 1 (Fig. 189),

$$\text{Total stress} = 282 \text{ tons.}$$

$$\text{Necessary area} = \frac{282}{6} = 47 \text{ square inches.}$$

Designed Area,

$$4 \text{ angle-irons } 3\frac{1}{2}'' \times 3\frac{1}{2}'' \times \frac{1}{2}'' = 13 \text{ square inches.}$$

$$1 \text{ top plate } 36'' \times \frac{1}{2}'' = 18 \quad ,,$$

$$2 \text{ side plates } 24'' \times \frac{3}{8}'' = 18 \quad ,,$$

$$\text{Total area} = \underline{49} \quad ,,$$

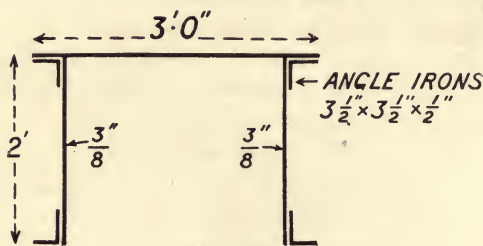


Fig. 189.

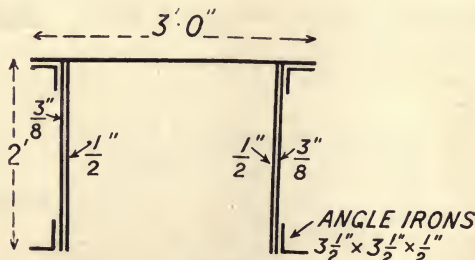


Fig. 190.

At Section 2 (Fig. 190),

Total stress = 426 tons.

Necessary area =  $\frac{426}{6} = 71$  square inches.

*Designed Area,*

4 angle-irons  $3\frac{1}{2}'' \times 3\frac{1}{2}'' \times \frac{1}{2}'' = 13$  square inches.

1 top plate  $36'' \times \frac{1}{2}'' = 18$  „

2 side plates  $24'' \times \frac{3}{8}'' = 18$  „

2 side plates  $24'' \times \frac{1}{2}'' = 24$  „



Total area = 73 „

NOTE. By adding vertical plates on the inside to get the necessary area, we do not alter the centre of gravity of the section. The same result may be got by adding plates on the outside between the angle-irons, in which case we get a constant distance between the side plates; which may be useful in designing the verticals, as they are generally fixed between the side plates.

### EXERCISES.

1. A rolled steel joist 16 inches deep, with flanges 6 inches wide and 1 inch thick, web  $\frac{3}{4}$  inch thick, supports a uniformly-distributed load of 2 tons per foot. If the span is 12 feet, determine the maximum tensile stress in the lower flange.

Ans.  $I = 847.5$ , stress = 5.5 tons per square inch.

2. Compare the strength to resist bending of the joist in Ex. 1 when it is placed upright like this,   $I$ , and when on its side like this,   $I$ .

3. A cast-iron girder 20 feet long is supported at the ends. The dimensions of the section are: Top flange, 3 inches by 1 inch; bottom flange, 8 inches by  $1\frac{1}{2}$  inches; web, 8 inches by  $1\frac{1}{4}$  inches. Find the moment of resistance, the greatest permissible compressive and tensile stresses being 7 and 2 tons per square inch respectively. Determine, also, the greatest safe load the girder will carry when uniformly distributed over its length.

4. A steel plate girder, 50 feet span, 5 feet deep, carries a uniform load of 1 ton per foot run, a load of 8 tons at centre, and two loads of 6 tons each at 10 feet on each side of the centre. Calculate the necessary areas of cross section of the flanges at the middle, and give suitable design for them.

Working tensile stress = 7 tons per square inch.

Working compressive stress = 6 tons per square inch.

Draw the diagram of bending moment.

5. Determine the safe load at the centre which a plate web girder of the following dimensions will carry: Span 40 feet; depth 5 feet; flanges, four plates each, 14 inches wide,  $\frac{1}{2}$  inch thick. The flange is attached to the web by two angles 4 inches by  $\frac{1}{2}$  inch. Reduce the area of each flange by two rivet holes  $\frac{3}{4}$  inch diameter connecting the flange and angles.

Working stress in flanges 5 tons per square inch.

6. Find the limiting span of a cast-iron pipe 12 inches external diameter, 1 inch thick, its weight being 100 lbs. per foot run. Stress not to exceed  $1\frac{1}{2}$  tons per square inch.

7. A circular tube 6 inches external diameter and  $\frac{1}{4}$  inch thick is supported at each end; span 5 feet. Find what load placed at the centre will produce a stress of 5 tons per square inch.

Find also the maximum intensity of shearing stress.

8. An oak beam 12 feet span, 9 inches wide, and 12 inches deep, carries a brick wall 9 inches thick. Find to what height the wall can be built with safety if the greatest permissible stress is 1000 lbs. per square inch; weight of brickwork 120 lbs. per cubic foot.

9. Compare the strength of an I section of depth 8 inches, breadth of each flange 3 inches, thickness of both flanges and web 1 inch, with that of a rectangular section of the same sectional area and depth.

*Ans.* I girder is the stronger in the ratio of 23 to 16.

10. Two beams of the same material are similarly loaded, one is round in cross section, the other is square. If the beams are of the same length and are of equal sectional area, which is the stronger?

*Ans.* The square beam is the stronger in the ratio of 1 to 0.847.

11. In the last example if the side of the square section is equal to the diameter of the circular section, compare the strengths of the two beams.

*Ans.* Strength of circular section = 0.589 of the square section.

12. A Pratt girder as in Fig. 181, span 104 feet divided into 8 bays, depth  $\frac{1}{10}$ th of the span; carries a dead load of 3.1 tons at each joint of the bottom chord, and a live load equivalent to 7.95 tons at the same joints. Determine the maximum stress in each member of the third panel.

*Ans.* Top member 103.62 tons compression.

Bottom member 82.90 tons tension.

Diagonal member 31.30 tons tension.

Vertical nearest support 19.5 tons compression.

„ „ centre 11.49 „ „

## CHAPTER VIII.

### DEFLECTION OF BEAMS.

#### 74. Deflection. Stiffness.

When a beam is loaded it becomes deflected or curved. The deflection is due to the bending moment, which causes longitudinal strains, of compression on one side of the neutral surface, and tension on the other side.

The elastic deflection of beams is important in all permanent engineering works. The beam or girder must not only be strong enough to sustain the loads, but must also be stiff enough to bear the loads without being strained beyond certain limits.

The two classes of beams which have generally to be considered in practice are :

- (1) Beams built up so as to have a uniform unit stress on each flange throughout. The *strength* and *depth* are uniform. This class includes all girders with parallel flanges.
- (2) Beams of uniform cross section throughout, where the value of  $I$  is constant for all sections, such as rectangular beams and rolled joists.

The stiffness of a girder is measured by the ratio

$$\frac{\text{maximum deflection}}{\text{span}}.$$

In practice this ratio is about  $\frac{1}{1200}$  to  $\frac{1}{1800}$  for long steel spans. For short girders and joists it is about  $\frac{1}{800}$ .

#### 75. Curvature.

The curvature of a circle is the reciprocal of its radius ; and of any curve it is the curvature of the circle which most nearly agrees with the curve ; or it may be defined as the angular change of the direction of the curve per unit of length.

#### 76. Curvature due to bending moment.

We have already shown that the bending moment

$$M = \frac{EI}{R} = \frac{f}{y} I,$$

where  $R$  is the radius of curvature of the portion of the bent beam considered.

**77. Uniform curvature. Beams of uniform strength and depth.**

From the above equation we see that

$$\frac{1}{R} = \frac{M}{EI} \text{ and } \frac{1}{R} = \frac{f}{Ey},$$

and it follows that a beam originally straight will bend into a circular arc if  $\frac{M}{I}$  or  $\frac{f}{y}$  is constant. This occurs (1) when a beam of uniform section is subjected to a uniform or constant bending moment, and (2) when a beam is so designed that the depth and flange stress are both uniform.

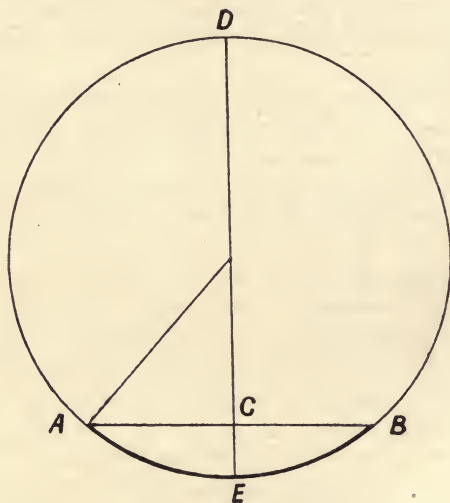


Fig. 191.

Let  $AB$  in Fig. 191 represent a beam of length  $L$  bent into a circular arc, and call  $u_1$  ( $CE$ ) the deflection at the centre, we have

$$CE \cdot CD = CA \cdot CB,$$

$$u_1 (2R - u_1) = \left(\frac{L}{2}\right)^2,$$

from which, since  $u_1$  is small and  $u_1^2$  may be omitted,

$$u_1 = \frac{L^2}{8R} = \frac{L^2 M}{8EI},$$

or, if  $d$  be depth of the girder and  $f$  the uniform flange stress,

$$u_1 = \frac{fL^2}{4dE}.$$

### 78. Curvature, slope, and deflection.

Let  $OPQ$  (Fig. 192) represent the curve assumed by the neutral surface after loading.  $OX$  is drawn horizontal. Take  $P$  and  $Q$ , two points on the curve *very close together*, the coordinates of which are  $x, u$ , and  $x + dx, u + du$  respectively.

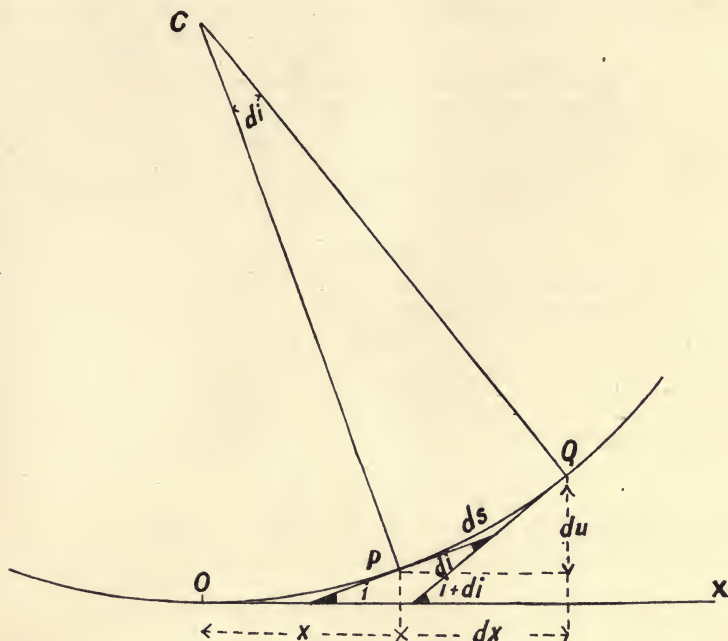


Fig. 192.

Call  $i$  the slope or inclination of the tangent at  $P$ , then  $(i + di)$  will be the inclination of the tangent at  $Q$ ,  $di$  being the angle between the tangents at  $P$  and  $Q$ , which is equal to the angle at the centre between the normals.

Now, the deflection being very small, and consequently  $dx$  sensibly equal to  $ds$ , the length of  $PQ$ ,

$$i = \frac{du}{dx}, \quad di = \frac{dx}{R}.$$

Therefore the curvature

$$\frac{1}{R} = \frac{di}{dx} = \frac{d^2u}{dx^2}. \quad \text{But } \frac{1}{R} = \frac{M}{EI},$$

$$\therefore \frac{d^2u}{dx^2} = \frac{M}{EI}.$$

The slope

$$\frac{du}{dx} = i = \int \frac{M}{EI} dx.$$

Deflection

$$u = \int i dx.$$

From these equations we determine the slope and deflection.

*Examples.*

(1) *Cantilever of length  $l$  of uniform section, loaded with a weight  $W$  at the free end (Fig. 193).*

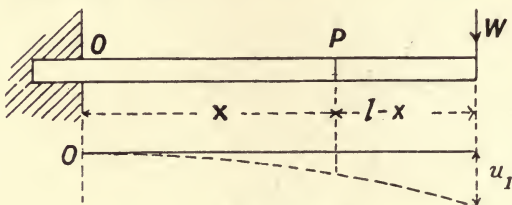


Fig. 193.

Take the origin at the fixed end of the beam and let  $x$  be the distance of a section from that end.

Then

$$M = W(l - x).$$

$$\therefore \frac{d^2u}{dx^2} = \frac{W}{EI}(l - x).$$

Integrating, we have, as  $E$  and  $I$  are constants,

$$i = \frac{du}{dx} = \frac{W}{EI}\left(lx - \frac{x^2}{2}\right) + C.$$

To find  $C$  we must know the slope at some one place.

Now, there is no slope at the fixed end, hence

$$\frac{du}{dx} = 0 \text{ where } x = 0; \text{ therefore } C = 0.$$

$$\text{Integrating again, } u = \frac{W}{EI}\left(\frac{1}{2}lx^2 - \frac{x^3}{6}\right) + C,$$

but

$$u = 0 \text{ when } x = 0, \therefore C = 0.$$

Deflection at any point of the beam

$$u = \frac{W}{EI}\left(\frac{1}{2}lx^2 - \frac{x^3}{6}\right).$$

Maximum deflection is at end, where  $x = l$ .

$$\therefore u_1 = \frac{Wl^3}{3EI}.$$

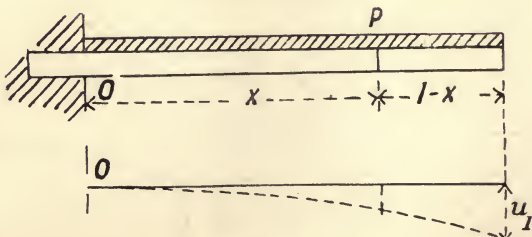


Fig. 194.

(2) *Cantilever of length  $l$  of uniform section uniformly loaded* (Fig. 194).

Let  $w$  be the load per unit of length, taking the origin as before at the fixed end, in order to eliminate the constants of integration,

$$M = w(l-x) \frac{(l-x)}{2} \\ = \frac{w}{2}(l-x)^2,$$

$$\therefore \frac{d^2u}{dx^2} = \frac{w}{2EI} (l^2 - 2lx + x^2).$$

Integrating 
$$i = \frac{du}{dx} = \frac{w}{2EI} \left( l^2x - lx^2 + \frac{x^3}{3} \right) + c;$$

but 
$$\frac{du}{dx} = 0 \text{ when } x = 0, \therefore c = 0.$$

To find the deflection, integrating again,

$$u = \frac{w}{2EI} \left( \frac{1}{2} l^2 x^2 - \frac{1}{3} l x^3 + \frac{1}{12} x^4 \right) + c_1;$$

now 
$$u = 0, \text{ when } x = 0, \therefore c_1 = 0;$$

hence the curve of the beam is

$$u = \frac{w}{24EI} (6l^2x^2 - 4lx^3 + x^4).$$

At the end, where  $x = l$ , we get the maximum deflection

$$u_1 = \frac{wl^4}{8EI} = \frac{Wl^3}{8EI},$$

if  $W = wl$ , the whole load on the beam.

In the two previous examples if the end  $O$  is imperfectly fixed, so that this end, instead of being horizontal, slopes at an angle  $\alpha$ , then

$$\frac{du}{dx} = \tan \alpha, \text{ when } x = 0$$

and

$$C = EI \tan \alpha.$$

(3) *Beam of uniform section supported at both ends, loaded with a single load  $W$  at the centre of the span* (Fig. 195).

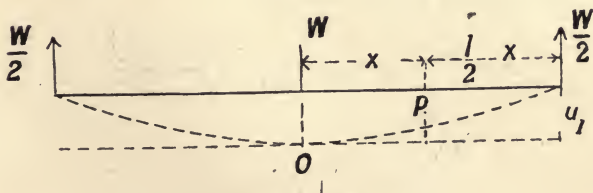


Fig. 195.

If we take the origin at the middle of the beam, the constant of integration is zero, since  $i = 0$  when  $x = 0$ ,

$$M = \frac{W}{2} \left( \frac{l}{2} - x \right);$$

$$\frac{d^2 u}{dx^2} = \frac{W}{2EI} \left( \frac{l}{2} - x \right).$$

Integrating, we get the slope at any point

$$i = \frac{du}{dx} = \frac{W}{2EI} \left( \frac{l}{2} x - \frac{x^2}{2} \right).$$

At the ends, where  $x = \frac{l}{2}$ , the slope

$$i_1 = \frac{Wl^3}{16EI}.$$

To find the deflection

$$u = \int i dx = \frac{W}{2EI} \left( \frac{lx^2}{4} - \frac{x^3}{6} \right);$$

and its greatest value, namely, the rise of the ends of the beam above the middle, is got by putting  $x = \frac{l}{2}$ ,

$$\therefore u_1 = \frac{Wl^3}{48EI}.$$

(4) *Beam of uniform section, supported at both ends, loaded with a uniformly-distributed load (Fig. 196).*

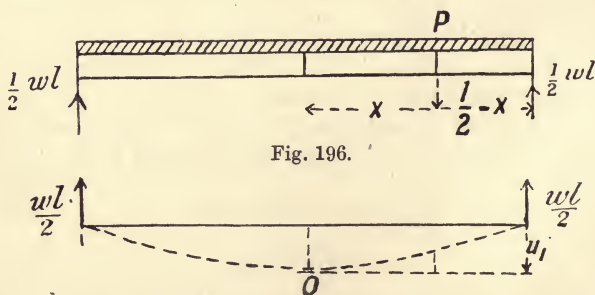


Fig. 196.

Take the origin at the centre as before, in order to make the constants of integration vanish, we have

$$\begin{aligned} M &= \frac{wl}{2} \left( \frac{l}{2} - x \right) - w \left( \frac{l}{2} - x \right) \frac{\left( \frac{l}{2} - x \right)}{2} \\ &= \frac{w}{2} \left( \frac{l^2}{4} - x^2 \right); \end{aligned}$$

so that

$$\frac{d^2 u}{dx^2} = \frac{w}{2EI} \left( \frac{l^2}{4} - x^2 \right).$$

Integrating, we get the slope at any point of the beam

$$i = \frac{du}{dx} = \frac{w}{2EI} \left( \frac{l^2 x}{4} - \frac{x^3}{3} \right).$$

Slope at ends where  $x = \frac{l}{2}$

$$i_1 = \frac{wl^3}{24EI}.$$

To find the deflection,

$$u = \int i dx = \frac{w}{2EI} \int_0^x \left( \frac{l^2 x}{4} - \frac{x^3}{3} \right) dx ;$$

$$\therefore u = \frac{w}{2EI} \left( \frac{l^2 x^2}{8} - \frac{x^4}{12} \right) ;$$

$u$  is greatest when  $x = \frac{l}{2}$ .

At the ends therefore

$$u_1 = \frac{5}{384} \frac{wl^4}{EI} = \frac{5}{384} \frac{Wl^3}{EI},$$

if  $W = wl$ , the total load.

This represents the rise at the ends, which is equal to the sag at the middle.

In the above cases, the maximum deflection may be expressed by the following general formulæ :

$$u_1 = \frac{Wl^3}{nEI} = \frac{Ml^2}{n_1 EI} = \frac{f_1 l^2}{n_1 E y_1},$$

where  $n$  and  $n_1$  are constants depending on the loading,

$l$  the length of the girder,

$y_1$  the distance of extreme fibre from neutral axis,

$f_1$  the intensity of stress at  $y_1$ .

For beams supported at the ends, we get the following values of  $n$  and  $n_1$ :

Load at centre,  $n = 48$  ;  $n_1 = 12$ .

Load uniform,  $n = \frac{384}{5}$  ;  $n_1 = \frac{48}{5}$ .

It will be seen later on that for beams *fixed* at the ends :

Load at centre,  $n = 192$  ;  $n_1 = 24$ .

Load uniform,  $n = 384$  ;  $n_1 = 16$ .

(5) A beam is supported at its ends, and loaded with a weight  $W$ , at a point  $F$ . Show that the deflection at  $F$  is  $\frac{Wa^2b^2}{3EI(a+b)}$ , where  $a, b$  are the distances from  $F$  to the points of support (Fig. 197).

Take the place where the weight acts at origin, and consider first the portion of the beam which lies to the right.

The reaction at the right support is  $\frac{Wb}{a+b}$ .

$CE$  is the tangent at  $O$ , and  $AD$  is drawn parallel to  $CE$ .

Now, if we consider the beam fixed at  $O$  and loaded at  $B$  with  $\frac{Wb}{a+b}$ , the bending moment at any section to the right of  $F$  is

$$\frac{Wb}{a+b} (a-x),$$

$$\therefore \frac{d^2u}{dx^2} = \frac{Wb}{EI(a+b)} \int_0^x (a-x) dx,$$

as in the case of a cantilever.

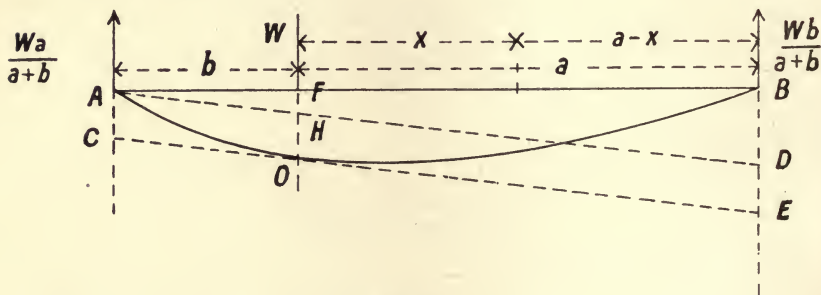


Fig. 197.

Integrating, 
$$i = \frac{du}{dx} = \frac{Wb}{EI(a+b)} \left( ax - \frac{x^2}{2} \right).$$

Integrating again, 
$$u = \frac{Wb}{EI(a+b)} \left( \frac{ax^2}{2} - \frac{x^3}{6} \right),$$

at  $x = a$ , 
$$u_1 = BE = \frac{Wba^3}{3EI(a+b)}.$$

Similarly, if we consider the portion of the beam on left of weight,

$$u_2 = AC = \frac{Wab^3}{3EI(a+b)}.$$

Deflection at  $F = FO = HO + FH = AC + FH$

$$= u_2 + \frac{b}{a+b} (u_1 - u_2)$$

$$= \frac{W}{3EI(a+b)} \left\{ ab^3 + \left( \frac{ba^3 - ab^3}{(a+b)} \right) b \right\}$$

$$= \frac{Wab^3}{3EI(a+b)}.$$

**79. Deflection of a beam supported at the ends, carrying a series of loads.**

*Example.*

A single-line railway bridge is carried by two main girders of 30 feet span, these girders support a continuous platform on which

the rails are laid. A locomotive on the bridge weighs 50 tons distributed upon three axles as follows, viz., 14 tons on the leading axle and 18 tons on each driving axle; the distances of the axles from the centre of support of main girders at one end being respectively 7 feet, 14 feet, and 22 feet. Find the maximum deflection.

Each main girder will carry half the weight of the locomotive as in Fig. 198,

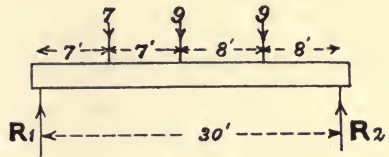


Fig. 198.

$$R_1 = 12.57 \text{ tons,}$$

$$R_2 = 12.43 \text{ tons.}$$

Segment 1 from the left support,

$$EI \frac{d^2u}{dx^2} = 12.57x,$$

$$EI \frac{du}{dx} = 6.28x^2 + C_1 \dots\dots\dots (A),$$

$$EIu = 2.09x^3 + C_1x + C_2 \dots\dots\dots (B).$$

Segment 2,

$$EI \frac{d^2u}{dx^2} = 12.57x - 7(x - 7),$$

$$EI \frac{du}{dx} = 6.28x^2 - 3.5(x - 7)^2 + C_1' \dots\dots\dots (C),$$

$$EIu = 2.09x^3 - \frac{3.5}{3}(x - 7)^3 + C_1'x + C_2' \dots\dots\dots (D).$$

Segment 3,

$$EI \frac{d^2u}{dx^2} = 12.57x - 7(x - 7) - 9(x - 14),$$

$$EI \frac{du}{dx} = 6.28x^2 - 3.5(x - 7)^2 - 4.5(x - 14)^2 + C_1'' \dots\dots\dots (E),$$

$$EIu = 2.09x^3 - \frac{3.5}{3}(x - 7)^3 - 1.5(x - 14)^3 + C_1''x + C_2'' \dots\dots\dots (F).$$

Segment 4,

$$EI \frac{d^2u}{dx^2} = 12.57x - 7(x - 7) - 9(x - 14) - 9(x - 22),$$

$$EI \frac{du}{dx} = 6.28x^2 - 3.5(x - 7)^2 - 4.5(x - 14)^2 - 4.5(x - 22)^2 + C_1''' \dots\dots\dots (G),$$

$$EIu = 2.09x^3 - \frac{3.5}{3}(x - 7)^3 - 1.5(x - 14)^3 - 1.5(x - 22)^3 + C_1'''x + C_2''' \dots\dots\dots (H).$$

To determine the values of the various constants, we have that at a section common to two adjacent segments, identical values for  $\frac{du}{dx}$  and for  $u$ , must be given by the equations for each segment. Thus

For  $x = 7$ .  $C_1 = C_1'$ ;  $C_2 = C_2'$ , from equations (A), (C); (B), (D).

$x = 14$ .  $C_1' = C_1''$ ;  $C_2' = C_2''$ , from equations (C), (E); (D), (F).

$x = 22$ .  $C_1'' = C_1'''$ ;  $C_2'' = C_2'''$ , from equations (E), (G); (F), (H).

Again, when  $x = 0$ ,  $u = 0$ , therefore by equation (B)  $C_2 = 0$ , hence

$$C_2' = C_2'' = C_2''' = 0,$$

when  $x = 30$ ,  $u = 0$ , therefore, equation (H),

$$30C_1''' = \frac{3 \cdot 5}{3} \times 23^3 + 1 \cdot 5 \times 16^3 + 1 \cdot 5 \times 8^3 - 2 \cdot 09 \times 30^3.$$

Therefore  $C_1''' = C_1'' = C_1' = -1182$ .

The maximum deflection will occur in the segment 3.

The abscissa,  $x_1$ , of this point is the value of  $x$  which makes  $\frac{du}{dx} = 0$  in equation (E). Hence

$$6 \cdot 28x_1^2 - 3 \cdot 5(x_1 - 7)^2 - 4 \cdot 5(x - 14)^2 - 1182 = 0.$$

Solving, we get  $x_1 = 14 \cdot 97$  ft.

Substituting this value of  $x_1$  for  $x$  in equation (F),

$$EIu_1 = -11256 \cdot 7.$$

Max. deflection is  $u_1 = \frac{11256 \cdot 7}{EI}$  foot,

where  $E$  is in tons per sq. foot, and  $I$  in foot units, that is (feet)<sup>4</sup>.

If  $E$ ,  $I$ , and  $u_1$  are in inch units,

$$u_1 = \frac{11256 \cdot 7 \times 12}{E \times 12^2 \times \frac{I}{12^4}} = \frac{11256 \cdot 7 \times 12^3}{EI} \text{ inches.}$$

## 80. Graphic method.

We have seen that the load, shearing force, and bending moment are connected by the relations

$$\frac{dF}{dx} = w; \quad \frac{dM}{dx} = F,$$

$$\frac{d^2M}{dx^2} = w,$$

where  $w$  is the load per unit of length.

Again, we have deduced the following relations between bending moment, slope, and deflection:

$$\frac{du}{dx} = i; \quad \frac{di}{dx} = \frac{d^2u}{dx^2} = \frac{M}{EI}.$$

Hence, if we know  $w$ , the loading of the beam, we can by integration obtain a succession of curves representing  $F$ ,  $M$ ,  $i$  and  $u$ .

From the above equations we see that the bending moment curve is the second integral of the load curve, and we know that it can be got from the load curve by means of a vector and link polygon. Similarly

the deflection curve, being the second integral of the  $\frac{M}{EI}$  curve, can be found from the bending moment curve by means of a vector and link polygon. Also, the ordinate of the slope curve at any point is equal to the area of the bending moment curve up to that point divided by  $EI$ ; and the deflection at any point is equal to the area of the slope curve up to that point.

### 81. Method of drawing deflection curve for unequal loading.

Having drawn the bending moment diagram, treat it as a load diagram; divide it up into narrow vertical strips (Fig. 199). Through the centre of each strip consider a force to act equal in magnitude to the area of the strip. Choose a pole  $O$ ; draw a vector and link polygon, and a line  $OX$  parallel to the closing line  $AB$  of the link polygon. Take a new pole  $O_1$  with  $O_1X$  horizontal, then a new link polygon is obtained with a horizontal base line, which gives the deflection on a certain scale.

*First*, to find the scale for bending moments.

Let

1 inch =  $m$  inches be the linear scale,

1 inch =  $n$  tons be the load scale,

Polar distance =  $h$  inches,

$M$  = bending moment,

$D$  = ordinate of bending moment diagram in inches.

Then 1 inch of ordinate of bending moment diagram represents

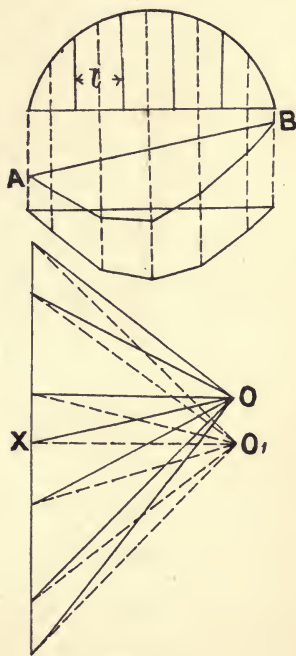
$$\frac{M}{D} = mnk \text{ inch-tons.}$$

*Secondly*, to find the scale for deflection curve.

Let  $l$  be the width of each of the narrow strips of bending moment diagram, and let  $k$  inches be the length of the new polar distance  $O_1X$ .

Now in the second vector polygon set down the loads as represented by the middle ordinate of each strip, consequently 1 inch on load scale now represents an area of  $lm \times mnk$ .

Thus the scale of deflection curve is  $\frac{m \times lm^2nh \times k}{EI}$  or, each inch of



-Fig. 199.

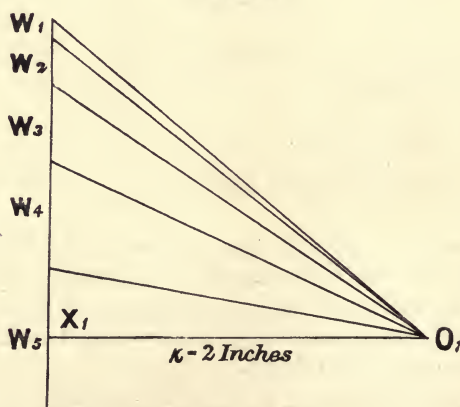
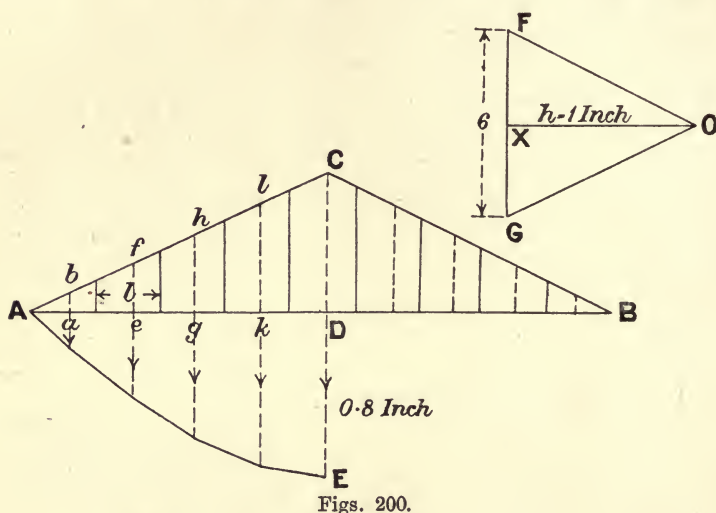
ordinate of deflection curve represents  $\frac{lm^3nhk}{EI}$  inches real deflection, the length units all being in inches.

*Example.*

For illustration of the method take the following simple case. A girder 15 feet long supported at the ends, carries a single load of 6 tons at the centre. If the moment of inertia of the cross section is 200 inch units, find the maximum deflection.  $E=12500$  tons per sq. inch.

Take  $\begin{cases} \text{a length scale of 1 inch} = 60 \text{ inches,} \\ \text{a load scale of 1 inch} = 6 \text{ tons.} \end{cases}$

Draw  $FG$  vertical to represent 6 tons, and take the polar distance  $OX(h)=1$  inch; join  $FO$  and  $GO$ , and draw the corresponding link



polygon, or bending moment diagram  $ACB$  (Fig. 200). The scale for this diagram is

$$1 \text{ inch} = mnk = 60 \times 6 \times 1 = 360 \text{ inch tons.}$$

Divide the bending moment diagram  $ABC$  (Fig. 200) into 9 vertical strips each  $\frac{1}{3}$ rd inch ( $l$ ) long, and at the centre of each strip suppose a force to act proportional to its area. Now on a vertical line (Fig. 201) measure off successively lengths equal to the middle ordinates  $ab, ef, gh, kl, CD$ . Take the pole  $O_1$ , making the polar distance

$$O_1X_1(k) = 2 \text{ inches,}$$

and draw the link polygon  $AEB$  (Fig. 200). The central ordinate  $DE$  (Fig. 200) gives the maximum deflection.

To get the scale,

$$\begin{aligned} 1 \text{ inch of ordinate} &= \frac{lm^3nhk}{EI} = \frac{\frac{1}{3} 60 \times 60 \times 60 \times 6 \times 2}{12500 \times 200} \\ &= \frac{864000}{2500000} = 0.34 \text{ inches deflection.} \end{aligned}$$

As the ordinate  $DE = 0.8$  inch,

$$\text{maximum deflection} = 0.8 \times 0.34 = 0.27 \text{ inches.}$$

The two following cases are worked on the same principles, but in a slightly different manner.

(1) *Beam of uniform cross section supported at both ends and loaded at the centre.*

The shearing-force diagram is a curve with ordinates of constant value, each  $= \frac{W}{2}$  (Fig. 202). The bending moment curve (Fig. 203) is

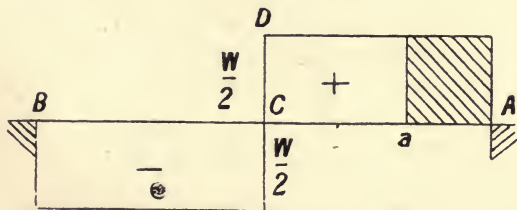


Fig. 202.

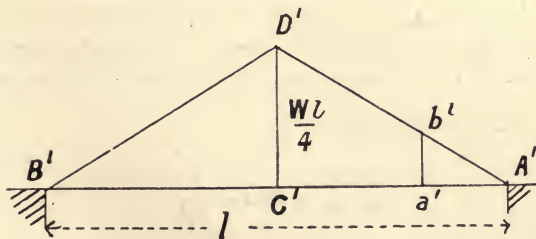


Fig. 203.

got from this by integration. The maximum ordinate  $C'D' = \text{area of the rectangle } AD$  (since  $M = \int Fdx = \frac{W}{2} \times \frac{l}{2} = \frac{Wl}{4}$ ).

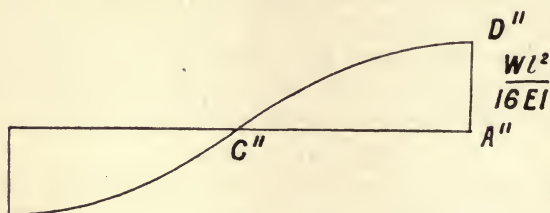


Fig. 204.

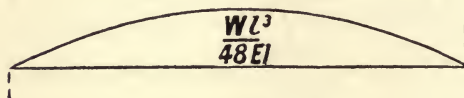


Fig. 205.

The curve of slope (Fig. 204) is got from the bending moment curve. The maximum slope at the end represented by the ordinate  $A''D''$  is the area of the triangle  $A'C'D'$  multiplied by  $\frac{1}{EI}$

$$= \frac{1}{2} \cdot \frac{Wl}{4} \cdot \frac{l}{2} \cdot \frac{1}{EI} = \frac{Wl^2}{16EI}.$$

The slope at the centre is zero.

To find the deflection curve (Fig. 205). The deflection at the ends = 0, and the maximum deflection at the centre is represented by the area of the semi-parabola  $A''C''D''$ ,

$$u_1 = \frac{2}{3} \cdot \frac{Wl^2}{16} \times \frac{l}{2} \times \frac{1}{EI} = \frac{1}{48} \frac{Wl^3}{EI}.$$

(2) Find the deflection of a uniform beam supported at the ends and loaded symmetrically with two equal loads (Fig. 206).

Fig. 207 is the bending moment curve, and Fig. 208 the curve of slopes. The slope at the centre is zero. The greatest slope is at the ends, and is equal to the area of the bending moment diagram between the end and centre multiplied by  $\frac{1}{EI}$

$$= \left( \frac{Wa^2}{2} + Wab \right) \frac{1}{EI},$$

as represented by the ordinate  $A'E'$ .

The maximum deflection is at the centre, and is equal to the area  $A'O'D'E'$  of the slope diagram multiplied by  $\frac{1}{EI}$

$$\begin{aligned}
 &= \left[ \frac{2}{3} \frac{W a^2}{2} \times a + W a^2 b + \frac{W a b^2}{2} \right] \frac{1}{EI} \\
 &= \frac{W a}{6EI} (2a^2 + 6ab + 3b^2),
 \end{aligned}$$

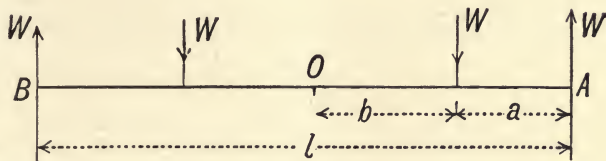


Fig. 206.

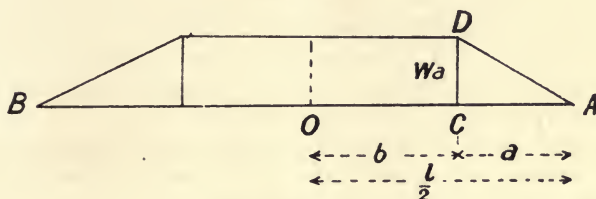


Fig. 207.

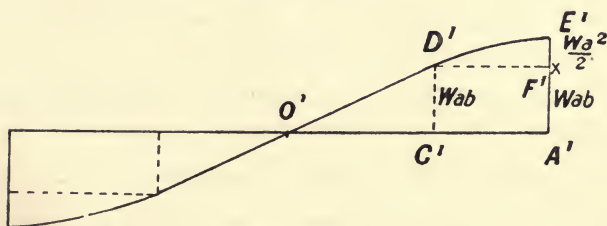


Fig. 208.

or, since

$$\begin{aligned}
 b &= \frac{l}{2} - a, \\
 u_1 &= \frac{W a}{24EI} (3l^2 - 4a^2).
 \end{aligned}$$

The student should apply the above principles to find the maximum deflection in the following cases, the cross section being assumed constant.

- (1) A horizontal beam supported at the ends ; (a) Loaded with a weight  $W$  dividing the span into two parts of length  $l_1$  and  $l_2$  ; (b) Loaded with a uniform load of intensity  $w$  per foot run.
- (2) A horizontal cantilever ; (c) Loaded with a weight  $W$  at the free end ; (d) Loaded with a uniform load of intensity  $w$  per foot run.

## 82. Fixed beams.

An encastré or fixed beam is one whose ends are fixed tangentially, so that they remain horizontal when the beam bends under the load (Fig. 209).

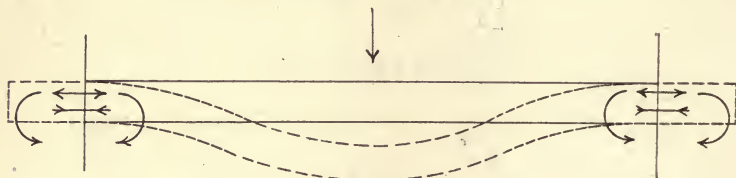


Fig. 209.

The bending will take place as in figure where the parts near ends are in "hogging" and the middle part in "sagging" curvature.

The direct stresses are distributed differently from those in supported beams. In the latter there is no direct stress at the points of support, the value of  $M$  there being  $=0$ . In the former the effect of the fixing is to produce a moment at the fixing, the upper fibres being in tension and the lower ones in compression.

The points where the curvature changes from hogging to sagging are called "points of contrary flexure" or "points of inflection"; at these points  $M=0$ , and there is no direct stress. We may, in fact, consider the whole beam to be made up, as far as the points of contrary flexure, of two cantilevers, one at each end, and a supported beam resting on the ends of those cantilevers.

## 83. Beam ends fixed horizontally. Uniformly loaded.

The loading being symmetrical, the fixing moments at the ends are equal and opposite. The reactions are each equal to  $\frac{wl}{2}$ .

Let  $M$  be the bending moment at any section of the beam, *if the ends are merely supported*; and let  $m$  be the fixing moments at the ends. Then the bending moment at the section is

$$M_x = m + M.$$

Thus at any point distant  $x$  from the end the bending moment

$$\begin{aligned} M_x &= m + \frac{wl}{2}x - wx \times \frac{x}{2} \\ &= m + \frac{wx}{2}(l-x) \dots\dots\dots(1). \end{aligned}$$

The slope

$$\begin{aligned} i &= \frac{1}{EI} \int M_x dx = \frac{1}{EI} \int \left( m + \frac{wx}{2}(l-x) \right) dx \\ &= \frac{1}{EI} \left( mx + \frac{wlx^2}{4} - \frac{wx^3}{6} \right) \dots\dots\dots(2). \end{aligned}$$

The constant of integration = 0, since  $i = 0$  when  $x = 0$ .

Again,  $i = 0$  when  $x = \frac{l}{2}$ ,

hence  $m \frac{l}{2} + \frac{wl^3}{16} - \frac{wl^3}{48} = 0$ ,

$$\therefore m = -\frac{wl^2}{12} \dots \dots \dots (3).$$

To find the bending moment at centre, in equation (1) give  $m$  its value from equation (3) and put  $x = \frac{l}{2}$ .

$$\begin{aligned} M \text{ at centre} &= -\frac{wl^2}{12} + \frac{wl}{2} \times \frac{l}{2} - \frac{wl^2}{8} = \frac{wl^2}{8} - \frac{wl^2}{12} \\ &= \frac{wl^2}{24}. \end{aligned}$$

The bending moment diagram is sketched in Fig. 210.

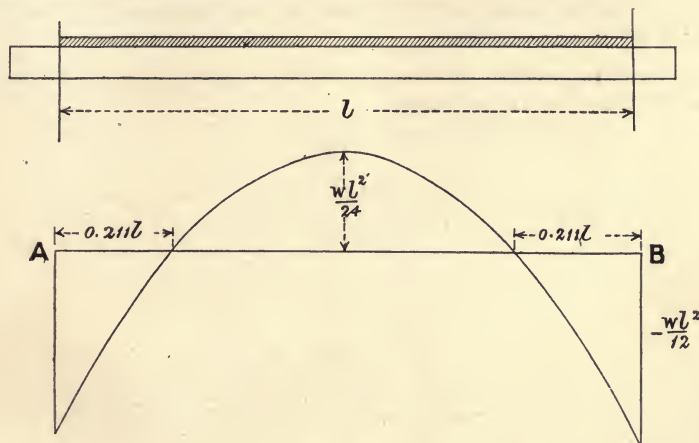


Fig. 210.

In a uniformly-loaded beam of span  $l$ , *simply resting on end supports*, the greatest bending moment is  $\frac{wl^2}{8}$ . In a fixed beam uniformly loaded the greatest bending moment (equation 3) is  $\frac{wl^2}{12}$ . The fixed beam is therefore stronger, so far as the maximum bending moment caused by a uniform load is concerned, in the proportion of 12 to 8 or 3 to 2.

To find the deflection of the fixed beam, uniformly loaded, we have from equation (2)

$$u = \int i dx = \frac{1}{EI} \int \left( -\frac{wl^2 x}{12} + \frac{wlx^2}{4} - \frac{wx^3}{6} \right) dx.$$

The maximum deflection at the middle, where  $x = \frac{l}{2}$ , is

$$u_1 = \frac{wl^4}{384EI} = \frac{Wl^3}{384EI},$$

which is only one-fifth of the deflection in a similarly-loaded beam supported at the ends.

From equation (1)

$$M_x = -\frac{wl^2}{12} + \frac{wx}{2}(l-x).$$

To find the points of inflection, put  $M_x = 0$ , which occurs when

$$x(l-x) = \frac{l^2}{6},$$

that is when

$$x = \frac{l}{2} \left( 1 \pm \frac{1}{\sqrt{3}} \right),$$

or  $x = 0.211l$  and  $x = 0.789l$  give the distances of the two points of inflection from either support.

**84. Beam fixed at both ends of uniform section carrying a load  $W$  at the middle.**

To find the bending moments at each support, and in the middle, also the deflection at the middle.

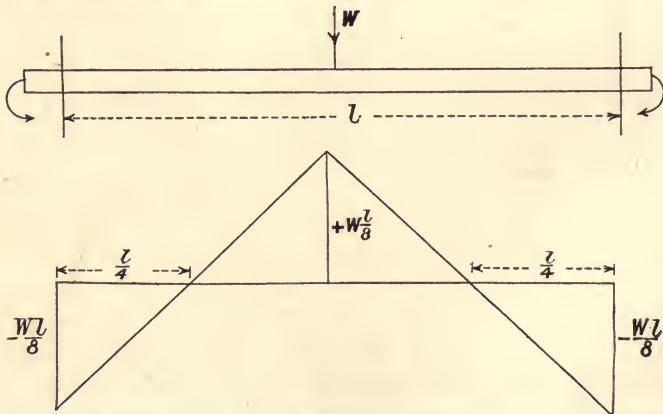


Fig. 211.

The vertical reaction at each end is equal to  $\frac{W}{2}$ .

At any point distant  $x$  from the end the bending moment is

$$M_x = m + \frac{W}{2}x \dots\dots\dots(4).$$

The slope  $i = \frac{1}{EI} \int M_x dx = \frac{1}{EI} \int \left( m + \frac{W}{2}x \right) dx$

$$= \frac{1}{EI} \left( mx + \frac{W}{4}x^2 \right).$$

The constant of integration must be zero, since  $i = 0$  when  $x = 0$ .

But  $i = 0$  when  $x = \frac{l}{2}$ , hence

$$m \frac{l}{2} + \frac{W}{4} \frac{l^2}{4} = 0,$$

$$\therefore m = -\frac{Wl}{8}.$$

Giving  $m$  its value in equation (4)

$$M_x = \frac{W}{2} x - \frac{Wl}{8} \dots\dots\dots(5);$$

at the centre, where  $x = \frac{l}{2}$ ,

$$M = \frac{Wl}{4} - \frac{Wl}{8} = \frac{Wl}{8}.$$

The diagram of bending moments has the form sketched in Fig. 211.

To find the points of inflection put  $M_x = 0$  in equation (5), this occurs when  $x = \frac{l}{4}$ .

To find the deflection

$$u = \int i dx = \frac{1}{EI} \int \left( -\frac{Wl}{8} x + \frac{W}{4} x^2 \right) dx$$

$$= \frac{1}{EI} \left( \frac{Wx^3}{12} - \frac{Wlx^2}{16} \right),$$

at the middle where the deflection is greatest  $x = \frac{l}{2}$ , and its amount is

$$u_1 = \frac{Wl^3}{192EI}.$$

**85. Uniform beam, uniformly loaded, fixed horizontally at one end, and supported at the other on the same level.**

The beam bends into the form shown in Fig. 212, there being a point of contrary flexure at  $E$ . Let  $P$  be the supporting force at  $B$ .

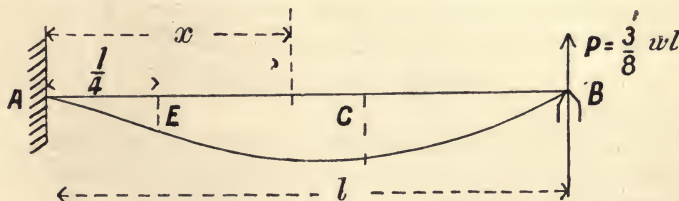


Fig. 212.

Take the origin at fixed end. Bending moment at any section

$$M_x = P(l-x) - \frac{1}{2}w(l-x)^2 \dots\dots\dots(6).$$

Slope

$$\begin{aligned} i = \frac{du}{dx} &= \frac{1}{EI} \int \left\{ P(l-x) - \frac{1}{2} w(l-x)^2 \right\} dx \\ &= \frac{1}{EI} \left\{ P \left( lx - \frac{x^2}{2} \right) - \frac{w}{2} \left( l^2 x - lx^2 + \frac{1}{3} x^3 \right) \right\}. \end{aligned}$$

The constant of integration = 0; since  $i = 0$  when  $x = 0$ . Integrating we get

$$u = \frac{1}{EI} \left\{ P \left( \frac{lx^2}{2} - \frac{x^3}{6} \right) - \frac{w}{2} \left( \frac{l^2 x^2}{2} - \frac{lx^3}{3} + \frac{x^4}{12} \right) \right\}.$$

Again constant = 0; since  $u = 0$  when  $x = 0$ .

But since at  $B$ , where  $x = l$ , we have  $u = 0$ ,

$$\therefore P \left( \frac{l^3}{2} - \frac{l^3}{6} \right) = \frac{w}{2} \left( \frac{l^4}{2} - \frac{l^4}{3} + \frac{l^4}{12} \right)$$

or

$$\frac{P}{3} = \frac{3wl}{24},$$

$$\therefore P = \frac{3}{8}wl.$$

So that  $\frac{3}{8}$ ths of the load is supported at  $B$ , and  $\frac{5}{8}$ ths at  $A$ .

To find the points of inflection, substitute in  $M_x$ , equation (6), the value of  $P$  just found, then

$$\begin{aligned} M_x &= (l-x) \left\{ \frac{3}{8}wl - \frac{wl}{2} + \frac{wx}{2} \right\} = (l-x) \left( \frac{wx}{2} - \frac{wl}{8} \right) \\ &= \frac{w}{2} (l-x) \left( x - \frac{l}{4} \right) \dots\dots\dots (7). \end{aligned}$$

Equating  $M_x$  to zero we get the point of contrary flexure at

$$x = \frac{l}{4}.$$

The moment at the fixed end is got by putting  $x = 0$  in equation (7)

$$m = -\frac{wl}{2} \times \frac{l}{4} = -\frac{wl^2}{8}.$$

By differentiating equation (7), and equating to zero, we find a maximum bending moment at  $x = \frac{5}{8}l$ , at this point  $C$

$$M_c = \frac{1}{2}w \times \frac{3}{8}l \times \frac{3}{8}l = \frac{9}{128}wl^2,$$

which is less than  $m$ .

Since  $m$  is equal to the maximum bending moment of a beam supported at the two ends and uniformly loaded, we see that a beam is *not made stronger by fixing one of the ends only*.

## 86. Resilience of a bent beam.

The work done in bending a short portion,  $dx$ , of a beam is  $\frac{1}{2}Mdi$ , where  $M$  is the bending moment, and  $di$  is the change of slope in the

length  $dx$ . Integrating from one end of the beam to the other, the whole work done is

$$\begin{aligned} & \frac{1}{2} \int M di \\ &= \frac{1}{2} \int \frac{M^2}{EI} dx, \text{ since } \frac{di}{dx} = \frac{M}{EI}. \end{aligned}$$

*Examples.*

1. Beam of uniform section subjected to a uniform bending moment, consequently uniform curvature.

$$\text{Work done} = \frac{M^2}{2EI} \times l;$$

$$\text{now } M = \frac{f_1}{y_1} I,$$

$$\text{therefore, } \text{Work done} = \frac{f_1^2 I l}{2 E y_1^2}.$$

$$\text{If the section is rectangular, } I = \frac{bd^3}{12} \text{ and } y_1 = \frac{d}{2},$$

$$\text{therefore, } \text{Work done} = \frac{f_1^2 b d l}{6 E},$$

$f_1$  being the intensity of stress at top or bottom,

$$\text{and, } \text{Work done} = \frac{f_1^2}{6 E} V, \text{ where } V = \text{volume,}$$

which is  $\frac{1}{3}$ rd the resilience of a piece subjected to a pull or push of uniform intensity  $f_1$ .

2. Beam of uniform section, supported at ends, loaded with a single load  $W$  at the middle.

The work done can be got by taking half the product of the load and the maximum deflection

$$= \frac{W}{2} \times \frac{W l^3}{48 EI} = \frac{W^2 l^3}{96 EI}.$$

In the case of a rectangular beam

$$M = \frac{f_1}{y_1} I; \quad I = \frac{bd^3}{12}; \quad y_1 = \frac{d}{2};$$

$$\text{Work done} = \frac{f_1^2 b d l}{18 E} = \frac{f_1^2}{18 E} \cdot V;$$

that is,  $\frac{1}{3}$ th of the resilience of piece subjected to a stress of uniform intensity  $f_1$ .

**87. Beam fixed at one end, supported at the other at the same level, and loaded at some intermediate point with a single load  $W$ .**

Here the bending moment is discontinuous.

$$\text{For } x < a, \quad M_x = P(l-x) - W(a-x).$$

$$\text{For } x > a, \quad M_x = P(l-x).$$

This makes the method of finding slope and deflection by integration inconvenient.

To find  $P$ , the pressure on the support, we can adopt the method of superposition of deflections.

Consider each load separately and add algebraically.

First consider  $W$ . See Fig. 213.

The portion  $ED$  of the curve  $AD$  is a straight line, and

$$DB = DC + CB = DC + EF = i_1 b + u_1,$$

where  $i_1$  is the slope of the beam at  $E$ ; but  $i_1$  of a beam fixed at one end and loaded at the other

$$= \frac{Wa^2}{2EI};$$

and

$$u_1 = \frac{Wa^3}{3EI};$$

$$\therefore DB = \frac{Wa^2b}{2EI} + \frac{Wa^3}{3EI} = \frac{Wa^2}{EI} \left\{ \frac{b}{2} + \frac{a}{3} \right\}.$$

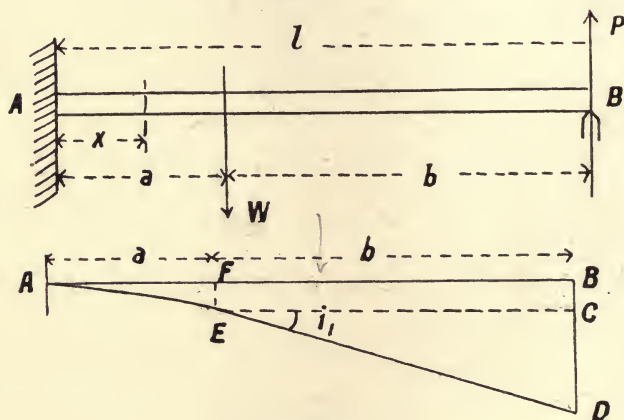


Fig. 213.

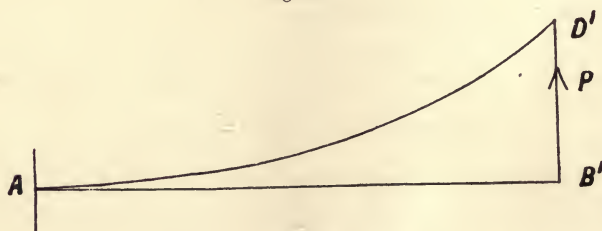


Fig. 214.

This would be the depression of the end  $B$  if  $W$  acted alone.

Secondly, consider  $P$  acting upwards at the free end of an encastré beam of length  $l$ ; then in Fig. 214 the deflection due to  $P$  is

$$B'D' = \frac{Pl^3}{3EI}.$$

Since the point  $B$  really keeps in the horizontal line, the deflection upwards due to  $P$  must be equal to the deflection downwards due to  $W$ , or

$$B'D' = BD;$$

$$\frac{Pl^3}{3EI} = \frac{Wa^2}{EI} \left\{ \frac{b}{2} + \frac{a}{3} \right\};$$

$$\therefore P = \frac{3Wa^2}{l^3} \left\{ \frac{b}{2} + \frac{a}{3} \right\}.$$

Knowing  $P$ , we can always find the bending moment at any point.

The point of contrary flexure can be found by equating the bending moment to zero.

As a special case of this example, suppose the weight at centre, then  $a = b = \frac{l}{2}$ , and

$$P = \frac{3Wa}{8a^3} \left\{ \frac{5}{6} \right\} = \frac{5}{16} W,$$

and the reaction at fixed end =  $\frac{11}{16} W$ .

This principle can be further applied to the case of a beam loaded with several weights,  $W_1, W_2$ , &c., dividing the length of the beam into sections,  $a_1, b_1; a_2, b_2$ , &c.; then

$$\begin{aligned} P &= \frac{3}{l^3} \sum Wa^2 \left\{ \frac{b}{2} + \frac{a}{3} \right\}; \\ &= \frac{1}{2l^3} \sum Wa^2 \{3b + 2a\}. \end{aligned}$$

**88. Beam supported at the ends and propped in the middle, uniformly loaded.** (Fig. 215.)

When a beam is loaded with several loads the deflection due to the whole is the sum of those due to each load taken separately.

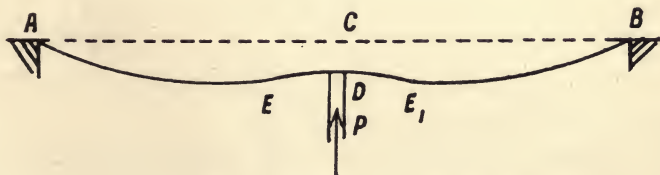


Fig. 215.

Hence the deflection in this case is the difference between the downward deflection due to the uniform load, and the upward deflection due to the thrust  $P$  of the prop.

Deflection at centre

$$\begin{aligned}
 &= \frac{5}{384} \frac{Wl^3}{EI} - \frac{Pl^3}{48EI} \\
 &= \frac{l^3}{48EI} \left( \frac{5}{8} W - P \right).
 \end{aligned}$$

Let us suppose the upward pressure such that this deflection = 0, or that the three supports are at the same level; we get

$$P = \frac{5}{8} W = \frac{5}{8} wl,$$

and consequently each of the supporting forces at the ends

$$= \frac{3}{16} W = \frac{3}{16} wl;$$

where, as before,  $w$  = load per foot run, and  $l$  the whole length of beam.

The bending moment at any point distant  $x$  from  $A$  is

$$M_x = \frac{3}{16} wlx - \frac{wx^2}{2} = \frac{wx}{2} \left( \frac{3}{8} l - x \right).$$

The points of contrary flexure are at  $E, E_1$ , where  $AE = \frac{3}{8} l = \frac{3}{4} AC$ .

Next, let us suppose the middle prop to be lower than the end supports. Assume the top of prop to be lower than the supports by  $\frac{1}{n}$ -th of the deflection of the beam when prop is removed, then

$$\frac{1}{n} \cdot \frac{5}{384} \frac{Wl^3}{EI} = \frac{5}{384} \frac{Wl^3}{EI} - \frac{Pl^3}{48EI};$$

that is,

$$P = \frac{5}{8} W \left( 1 - \frac{1}{n} \right).$$

### EXERCISES.

1. A beam uniformly loaded rests on three supports, two at the ends and one at the middle. Find how much the middle support must be lower than the end supports in order that the pressures on the three supports shall be equal.

$$\text{Ans. } \frac{7}{1152} \frac{Wl^3}{EI}.$$

2. A uniformly-loaded beam rests on three supports distant apart  $l$  and  $2l$ ; the centre prop is  $\frac{l}{n}$  below the level of the two end supports. Find the reaction on centre prop.

3. A beam of rectangular section, depth 8 inches, breadth 3 inches, 10 feet long, is supported horizontally at the ends and loaded with a weight of 5 tons uniformly distributed. Find the deflection, assuming  $E = 12000$  tons per square inch.

4. A beam 20 feet span supported at both ends carries loads of 2 tons, 4 tons, 1 ton and 3 tons at points 3 feet, 8 feet, 12 feet, and

17 feet respectively from the left support. Find graphically the maximum deflection. Assume  $I=300$ ;  $E=12000$  tons per square inch.

5. A wooden joist of a floor carries a uniform load of 0.2 ton per foot run. If the span is 20 feet, determine the scantling of the joist if the deflection is limited to  $\frac{5}{8}$  inch; the maximum stress being  $\frac{1}{2}$  ton per square inch. Assume  $E=700$  tons per square inch.

*Ans.* 14 inches deep by  $7\frac{3}{8}$  inches wide.

6. If in the last exercise, the depth of the joist is limited instead of the stress; the maximum depth allowable being 13 inches. Determine the necessary scantling.

*Ans.* 13 inches by 9 inches.

7. A girder 30 feet span supported at the ends carries two loads, one of 7 tons at 6 feet, the other of 12 tons at 18 feet from the left support. Find the maximum deflection.

$$\text{Ans. } u_1 = \frac{8608}{EI} \text{ foot.}$$

8. A girder 50 feet span supported at the ends is loaded as follows :

7 tons at 10 feet from left support.

3    "    15    "    "

5    "    21    "    "

4    "    30    "    "

Find the maximum deflection.

$$\text{Ans. } u_1 = \frac{39000}{EI} \text{ foot.}$$

9. A fir beam 12 feet span, 10 inches deep, 5 inches wide, supported at the ends, carries a uniform load of 5 cwt. per foot of span. Find the maximum deflection.  $E=700$  tons per square inch.

*Ans.* 0.4 inch.

10. A rectangular wooden beam 20 feet span, 14 inches deep, carries a load of 2 tons at the centre with a maximum stress of 1000 lbs. per square inch. Find the maximum deflection, and the breadth of beam.

$$\text{Ans. } \frac{685714}{E \text{ in lbs. per square inch}} \cdot 8.23 \text{ inches.}$$

11. A beam 20 feet span is fixed at the ends, carries a load of 5 tons in the centre, and loads of 2 tons each at 5 feet from the ends. Construct the bending moment diagram. Give the value of the maximum bending moment, and the position of the points of inflection.

*Ans.* 20 foot-tons. 4.4 feet from the ends.

12. Determine the ratio between the deflection of a girder of uniform  $I$  cross section, and one of uniform strength, both girders

being designed for the same span, with the same uniform depth, and the same maximum working stress per square inch of flange section under a central load.

*Ans.* 2 : 3.

13. In last exercise if the load is uniformly distributed, determine the relative deflections for the same maximum stress.

*Ans.* 5 : 6.

14. A horizontal beam of uniform section, whose moment of inertia is  $I$ , and whose total length is  $2l$ , is supported at the centre, one end being anchored down to a fixed abutment. Neglecting the weight of the beam, suppose it to be loaded at the other end with a single weight  $W$ . Find an expression for the vertical deflection of that end below its unstrained position.

*Ans.*  $\frac{7Wl^3}{12EI}$ .

15. A horizontal beam fixed at both ends carries a weight  $W$  at a point which divides the span into two portions  $a$  and  $b$ . Find the deflection at the point of application of the weight, and the work done in bending the beam.

*Ans.*  $\frac{W}{3EI} \left( \frac{ab}{a+b} \right)^3$ ;  $\frac{W^2}{6EI} \left( \frac{ab}{a+b} \right)^3$ .

16. A girder 42 feet span, supported at the ends, carries two concentrated loads, one of 6 tons at 7 feet, and one of 10 tons at 28 feet from the left support. Find the maximum deflection.

*Ans.*  $u_1 = \frac{17537}{EI}$  foot.

$E$  in tons per square foot,  $I$  (feet)<sup>4</sup>.

17. A girder 100 feet span carries three loads of 20 tons placed at 20, 40, and 60 feet respectively from the left support. Determine the maximum deflection.

*Ans.* The maximum deflection occurs at a section 48.6 feet from the left support, its value is

$u_1 = \frac{1024312 \times 12^3}{EI}$  inches.

18. A cross girder of a railway bridge carrying a double line of way is 27 feet span between the main girders, and the four rails are carried by it at 3 feet and 8 feet respectively from the centre of span on each side. The maximum load at each of these four points when two trains are on the bridge may be taken as 8 tons. Find the maximum deflection.

*Ans.* The maximum deflection occurs at the centre of span,

$u_1 = \frac{9899 \times 12^3}{EI}$  inches.

## CHAPTER IX.

### BENDING AND DIRECT STRESS.—NON-AXIAL LOADS.—STRESS AT A PLANE JOINT.—MASONRY STRUCTURES.

#### 89. Uniform stress. Uniformly-varying stress.

We know that in bent beams the stress on a cross section is a *uniformly-varying stress*. The following are also examples of this state of stress:—Ties and struts where the load, although parallel to, does not act along the axis of the piece—that is, the load is non-axial; also masonry piers and arches, where the line of action of the resultant thrust does not pass through the centre of gravity of the joint.

**UNIFORM STRESS.** When a load is applied along the axis of a piece it produces a uniform distribution of stress over the surface of a cross section; the intensity of stress at all points of the surface is uniform and constant; and the resultant of the stress on the surface acts at a point called the *centre of stress*, which in this case coincides with the centre of area.

**UNIFORMLY-VARYING STRESS.** If, on the other hand, the centre of stress of the cross section does not coincide with the centre of area, then the distribution of stress over the surface is unequal, and it is assumed that the stress is a *uniformly-varying* one—that is, the intensity of stress at any point in the section varies directly as the distance of that point from a fixed line in the plane of the section. This line is called the *neutral axis of the stress*.

Thus, if ordinates be drawn at right angles to the stressed surface  $AB$  (Figs. 216 and 217), each representing the intensity of stress at the point on which it is erected, the locus of the extremities of these ordinates will be in a plane  $CD$ , which will be parallel or inclined to the surface  $AB$ , according as the intensity of stress is uniform (Fig. 216) or uniformly varying (Fig. 217). The volume of the cylinder represents the total amount of stress. The line through  $E$  at right angles to the plane of the paper is the neutral axis of the uniformly-varying stress. It is evident from Fig. 217 that in the case of a uniformly-varying stress the resultant falls on one side of the centre of area of the surface  $AB$ .

When, as in Fig. 218, the neutral axis of the uniformly-varying stress falls outside the surface  $AB$ , then the stress is of one sign all

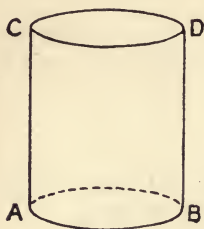


Fig. 216.

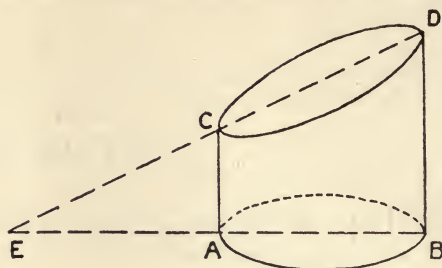


Fig. 217.

over the surface. If the neutral axis, as in Fig. 219, falls within the surface  $AB$  it divides the surface into two parts, on one of which there is tension, and on the other compression. If the neutral axis passes through the centre of area of the surface, then (as in bent beams) the total tension on one side is equal to the total compression on the other, and the resultant of the stress is a couple.

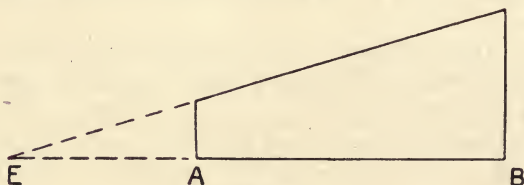


Fig. 218.

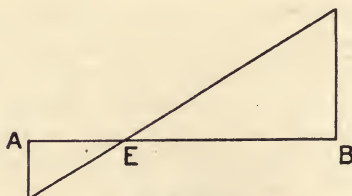


Fig. 219.

### 90. Stress on a section or joint, the load being non-axial.

Let  $AB$  (Fig. 220) represent the trace of a surface on a plane at right angles to it,  $O$  being that of a line through its centre of area. Let  $F$  be the resultant force normal to the surface, its line of action intersecting  $AB$  in  $M$ .  $F$  is also the resultant *internal stress* developed at  $AB$ .

Let  $OM = x_0$ ; that is, the distance of centre of stress from centre of area.

Let  $x_1$  and  $x_2$  be the distances from  $O$  of  $B$  and  $A$  respectively.

Let  $S$  = area of surface  $AB$ .

Let  $f_1$  and  $f_2$  be the extreme intensities of stress at  $B$  and  $A$ .

Let  $f_0$  be the intensity of stress at centre of area of  $AB$ .

Let  $I$  be the moment of inertia of the surface about an axis through  $O$  at right angles to the plane of the figure.

Then the stress represented by  $ABCD$  (Fig. 220) may be considered as made up of two parts, viz. :—(a) A uniform stress  $AGHB$ , due to

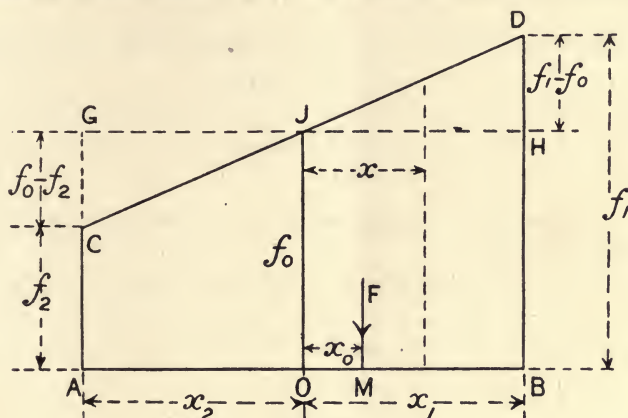


Fig. 220.

a load equal to  $F$  acting at the centre of area  $O$ , the intensity of which is  $f_0 = \frac{F}{S}$ ; and (b) a uniformly-varying stress  $GCHD$ , due to a bending moment  $M = F \cdot x_0$ , represented by the triangular figures  $HDJ$  (compressive) and  $GCJ$  (tensile), compressive and tensile stresses being regarded as of opposite signs. Compressive and tensile stresses are regarded as positive and negative respectively.

The intensity of this uniformly-varying stress on any line distant  $x$  from  $O$  is

$$\frac{M \cdot x}{I} = \frac{F x_0 x}{I}.$$

Adding (a) and (b) we get :

The intensity of stress at edge  $B$

$$f_1 = f_0 + \frac{F x_0 x_1}{I} \dots \dots \dots (1)$$

$$= \frac{F}{S} \left( 1 + \frac{S x_0 x_1}{I} \right) \dots \dots \dots (2).$$

The intensity of stress at edge  $A$

$$f_2 = f_0 - \frac{F x_0 x_2}{I} \dots \dots \dots (3).$$

$$= \frac{F}{S} \left( 1 - \frac{S x_0 x_2}{I} \right) \dots \dots \dots (3).$$

We have thus a uniform stress all over the section or joint  $= \frac{F}{S}$  as well as a uniformly-varying stress (compressive on one side of  $O$ , and tensile on the other side), which the bending moment  $Fx_0$  produces. If the resultant of the loading forces is a force  $R$  inclined to the section or joint, then  $F$  in the above equations is the component of  $R$  normal to  $AB$ .

In symmetrical sections  $x_2 = x_1$ ; and equations (2) and (3) become

$$f_1 = \frac{F}{S} \left( 1 + \frac{Sx_0x_1}{I} \right) \dots\dots\dots(4),$$

$$f_2 = \frac{F}{S} \left( 1 - \frac{Sx_0x_1}{I} \right) \dots\dots\dots(5).$$

**91. To find the limiting value of  $x_0$  without reversing the sign of the stress.**

It is necessary, especially in the case of bed joints of masonry structures, to limit the value of  $x_0$ , in order to ensure that at no part of the joint will the stress be tensile. To fulfil this condition put

$$f_2 = 0;$$

then

$$x_0 = \frac{I}{Sx_1} \dots\dots\dots(6).$$

#### RECTANGULAR JOINT.

For a rectangular joint in masonry whose sides are  $t$  and  $l$  (Fig. 221) we have

$$I = \frac{lt^3}{12}, \quad S = lt, \quad x_1 = \frac{t}{2}.$$

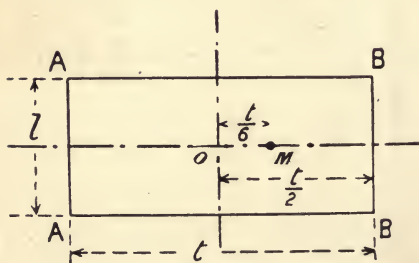


Fig. 221.

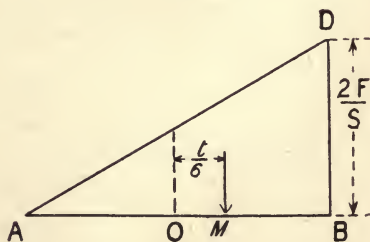


Fig. 222.

Substituting in equation (6)

$$\text{Limiting value of } x_0 = \frac{t}{6} \dots\dots\dots(7),$$

as  $x_0$  may be on either side of  $O$ , we have from equation (7) that the resultant thrust must fall within the middle third of the joint in order that there may be no tensile stress on any part of the joint.

When  $x_0 = \frac{t}{6}$ , the value of the stress varies from  $2 \left( \frac{F}{S} \right)$  at the edge nearest the resultant, to zero at the opposite edge (Fig. 222).

#### CIRCULAR SECTION.

Let  $r$  = radius. Then, for this section,

$$S = \pi r^2, \quad I = \frac{\pi r^4}{4}, \quad x_1 = r.$$

Therefore  $x_0 = \frac{r}{4} \dots \dots \dots (8);$

hence the limit of deviation of the point of application of the resultant from the centre to ensure stress of the same sign all over the circular section is

$$x_0 = \frac{r}{4} = \frac{D}{8}.$$

In a *hollow circular section* of outside diameter  $D$  and inside diameter  $D_1$  the limit of deviation is

$$x_0 = \frac{D^2 + D_1^2}{8D}.$$

**91A. Another method of determining the extreme intensities of stress  $f_1$  and  $f_2$  on a rectangular joint.**

The position of the centre of stress  $M$  is sometimes given by its distance from the nearest edge of the section or joint.

Let  $F$  = resultant normal pressure.

„  $t$  = length of joint  $AB$ .

„  $d$  = distance of the centre of stress  $M$  from the edge  $B$ .

„  $f_1$  and  $f_2$  be the maximum and minimum stresses at  $B$  and  $A$  respectively.

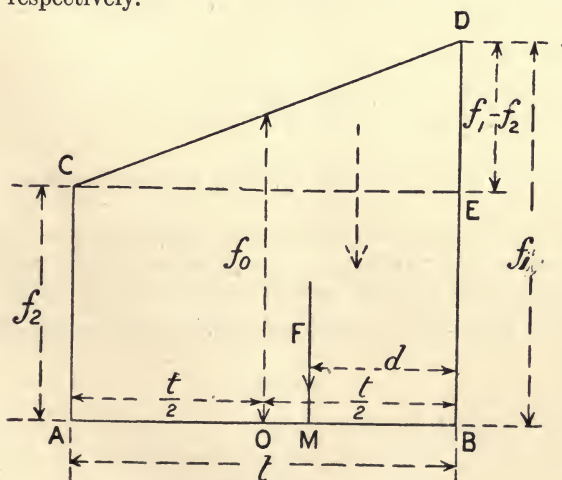


Fig. 223.

Let the ordinates  $BD$  and  $AC$  (Fig. 223) be drawn to represent  $f_1$  and  $f_2$  respectively. Then the trapezoid  $ABDC$  represents the total stress on the joint. Draw  $CE$  parallel to  $AB$ .

*Let the width of the joints at right angles to the paper be unity.*

Then the total stress on the joint is made up of two parts, viz.

$ABEC = f_2 t$  acting at  $O$ , and  $EDC = \frac{(f_1 - f_2)}{2} t$  acting at a distance  $\frac{t}{3}$  from  $B$ .

Take moments round  $B$ ,

$$Fd = f_2 t \times \frac{t}{2} + \frac{(f_1 - f_2)}{2} t \times \frac{t}{3} \\ = \frac{f_2 t^2}{3} + \frac{f_1 t^2}{6} \dots \dots \dots (9).$$

But  $F = \frac{1}{2} (f_1 + f_2) t$ ;

Therefore  $f_2 = \frac{2F}{t} - f_1 \dots \dots \dots (10).$

Substituting in (9) for  $f_2$  its value in (10),

$$\text{Maximum intensity of stress} = f_1 = \frac{2F}{t} \left( 2 - \frac{3d}{t} \right) \dots \dots \dots (11).$$

Similarly from equations (9) and (10),

$$\text{Minimum intensity of stress} = f_2 = \frac{2F}{t} \left( \frac{3d}{t} - 1 \right) \dots \dots \dots (12).$$

In equation (11)  $\frac{F}{t}$  is the value of the average stress-intensity on the base, that is,  $f_0$ .

Therefore  $f_1 = 2f_0 \left( 2 - \frac{3d}{t} \right)$ ;

or  $d = \frac{t}{3} \left( 2 - \frac{f_1}{2f_0} \right) \dots \dots \dots (13).$

Equation (13) gives the value of  $d$ , the distance of the centre of pressure from the extremity of base, where  $f_1$  is the maximum stress-intensity, and  $f_0$  is the average stress-intensity on the base.

To get the limiting value of  $d$  so that there shall be no tension on the joint, put

$$f_2 = 0 ;$$

to fulfil this condition we must have

$$d = \frac{t}{3}.$$

92. Given a non-axial load acting on a section, to determine where the stress changes sign. (For this solution I am indebted to Prof. G. M. Minchin, F.R.S.)

Fig. 224 (A) represents the diagram of stress on the joint.

In Fig. 224 (B) let  $AHBEA$  represent a normal cross section of a pillar or column, and let a non-axial load  $F$  act on the line  $PQ$ ; the centre of pressure being at  $M$ . Let  $O$  be the centre of area of the cross section, and  $KL$  the neutral axis of stress.

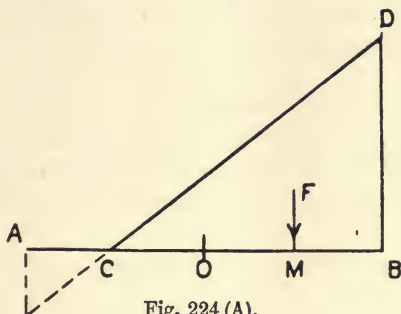


Fig. 224 (A).

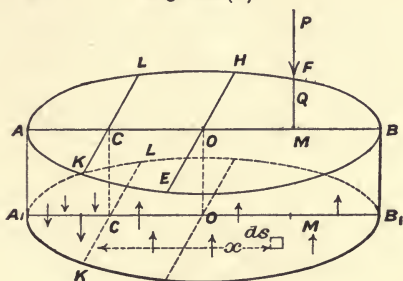


Fig. 224 (B).

Take a parallel normal cross section  $A_1B_1$  at a *very small* distance below  $AB$ , and consider the equilibrium of the portion of column between  $AB$  and  $A_1B_1$ .

The resultant upward pressure on  $A_1B_1$  must be exactly equal and opposite to  $F$ .

Consider a small element of area  $dS$  of  $A_1B_1$  distant  $x$  from the neutral axis through  $C$ .

The pressure on this element  $= \lambda x dS$  where  $\lambda$  is a constant.

Therefore, the total pressure on the area  $A_1B_1$  is

$$\begin{aligned} F &= \lambda \int x dS \\ &= \lambda S \cdot CO. \end{aligned}$$

Again, taking moments about  $KL$ ,

$$F \cdot MC = \lambda \int x^2 dS = \lambda S k_c^2$$

where  $k_c$  is the radius of gyration of the area about the axis  $KL$ .

Substituting for  $F$  its value above,

$$MC \times CO = k_c^2 = k_0^2 + \overline{CO}^2,$$

or

$$(MO + CO) CO = k_0^2 + \overline{CO}^2;$$

therefore

$$MO \times CO = k_0^2 \dots\dots\dots(14),$$

$k_0$  being the radius of gyration of the area about the axis  $EH$  through  $O$  the centre of area.

From (14) we see that the points  $M$  and  $C$  are related as the centre of oscillation and centre of suspension in a compound pendulum whose centre of gravity is  $O$  and radius of gyration  $k_0$ .

### Examples.

1. Determine the greatest and least intensities of compressive stress on a normal cross section of a rectangular pillar, the width of which is 4 feet and the breadth 2 feet, due to a load of 250 tons. The line

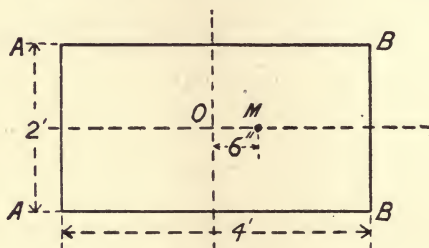


Fig. 225.

of action of the load is 6 inches from the axis of pillar, in the central plane parallel to the width (Fig. 225).

$$S = 8 \text{ square feet}; I = \frac{2 \times 4^3}{12} = \frac{32}{3}; x_0 = \frac{1}{2} \text{ foot.}$$

$$\text{Max. stress } f_1 = \frac{F}{S} \left( 1 + \frac{Sx_0x_1}{I} \right)$$

$$= \frac{250}{8} \left( 1 + \frac{8 \times \frac{1}{2} \times 2}{\frac{32}{3}} \right)$$

$$= \frac{250}{8} \left( 1 + \frac{24}{32} \right) = 54.7 \text{ tons per square foot compressive.}$$

$$\text{Min. stress } f_2 = \frac{250}{8} \left( 1 - \frac{24}{32} \right) = 7.8 \quad \text{,,} \quad \text{,,} \quad \text{,,} \quad \text{,,}$$

2. Determine the maximum intensities of compressive and tensile stress on the cross section, 6 inches square, of a wooden post under a vertical load of 7.5 tons acting at  $1\frac{1}{4}$  inches from the axis of the post in a central plane parallel to one side of the post.

$$S = 36 \text{ square inches}; x_0 = \frac{5}{4} \text{ inches}; F = 7.5 \text{ tons}; I = \frac{6^4}{12} = 108.$$

$$\begin{aligned}\text{Max. compressive stress} &= \frac{F}{S} \left( 1 + \frac{Sx_0x_1}{I} \right) \\ &= \frac{7.5 \times 2240}{36} \left( 1 + \frac{5}{4} \right) \text{ lbs. per square inch} \\ &= 1,050 \text{ lbs. per square inch.}\end{aligned}$$

$$\begin{aligned}\text{Max. tensile stress} &= \frac{7.5 \times 2240}{36} \left( 1 - \frac{5}{4} \right) \text{ lbs. per square inch} \\ &= -116.8 \text{ lbs. per square inch.}\end{aligned}$$

3. A wrought-iron bar of rectangular section, 3 inches by 1 inch, transmits a tensile force of 5 tons. The bar is cranked so that the line of action of the load, though parallel to the axis of the bar, coincides with the middle of one of the smaller sides. Determine the maximum intensities of stress in a normal cross section.

$$S = 3 \text{ square inches ; } F = -5 \text{ tons ; } x_0 = \frac{3}{2} \text{ inches ;}$$

$$f_1 = \frac{F}{S} \left( 1 + \frac{Sx_0x_1}{I} \right) = \frac{-5}{3} (1 + 3)$$

$$= -6.6 \text{ tons per square inch (tensile);}$$

$$f_2 = -\frac{5}{3} (1 - 3)$$

$$= +1.66 \text{ tons per square inch (compressive).}$$

4. A masonry dam has a horizontal base 115 feet wide. It retains a depth of water of 150 feet. Assume that the weight of one foot in length of the dam is 500 tons, and that the resultant acts at 45 feet from the right-hand edge of base. Determine the maximum intensities of vertical stress on the base (Fig. 226).

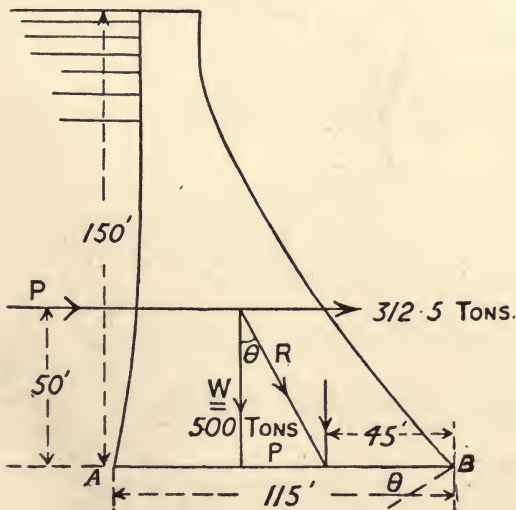


Fig. 226.

Consider one foot in length of the dam.

The total water pressure is

$$\frac{wh^2}{2} = \frac{1}{36} \frac{150^2}{2} = 312.5 \text{ tons.}$$

*Note.* One cubic foot of water weighs  $\frac{1}{36}$ th of a ton. The horizontal component of resultant is 312.5 tons. The vertical component of resultant is 500 tons.

Substituting in the formulæ (11) and (12),

$$\begin{aligned} \text{Max. intensity of stress } f_1 &= \frac{2W}{t} \left( 2 - \frac{3d}{t} \right) \\ &= \frac{2 \times 500}{115} \left( 2 - \frac{3 \times 45}{115} \right) \\ &= 7.18 \text{ tons per square foot.} \end{aligned}$$

$$\begin{aligned} \text{Min. intensity of stress } f_2 &= \frac{2W}{t} \left( \frac{3d}{t} - 1 \right) \\ &= \frac{2 \times 500}{115} \left( \frac{3 \times 45}{115} - 1 \right) \\ &= 1.5 \text{ tons per square foot.} \end{aligned}$$

These are the values of the *vertical* intensities at the outer edges of the horizontal base *B* and *A* respectively. The *mean* intensity of stress is  $\frac{8.62}{2} = 4.34$  tons per square foot.

The *maximum intensity of pressure* is on a plane at right angles to the resultant *R*.

The resultant pressure =  $\sqrt{500^2 + 312.5^2} = 589$  tons.

The maximum intensity of stress on a section at right angles to the resultant

$$= \frac{f_1}{\cos \theta};$$

$$\text{where} \quad \cos \theta = \frac{W}{\sqrt{W^2 + P^2}} = \frac{500}{589}.$$

Hence,

$$\begin{aligned} \text{The maximum stress-intensity at } B &= 7.18 \times \frac{589}{500} \\ &= 8.48 \text{ tons per square foot.} \end{aligned}$$

$$\begin{aligned} \text{The minimum stress-intensity at } A &= 1.5 \times \frac{589}{500} \\ &= 1.75 \text{ tons per square foot.} \end{aligned}$$

*Note.* When the resultant pressure is inclined to the horizontal, the *maximum* intensity of stress is found not by taking the normal component of the resultant pressure acting at the centre of pressure, but by considering a joint at right angles to the resultant *R*. This.

joint makes an angle  $\theta$  with the horizontal. The maximum intensity of compressive stress is therefore, for

$$d > \frac{t}{3}; \quad f_1 = \frac{2R}{t \cos \theta} \left( 2 - \frac{3d}{t} \right)$$

and not

$$f_1 = \frac{2W}{t} \left( 2 - \frac{3d}{t} \right),$$

$W$  being the normal component of  $R$ . The stress is greater in the ratio of 1 to  $\cos^2 \theta$ .

### 93. Stability of masonry structures.

In order to illustrate the application of the formulæ in this chapter we will apply them to a few simple cases of masonry structures. For a further treatment of "dams," see *Principles of Waterworks Engineering*, by Tudsbery and Brightmore, from which one or two examples have been taken.

The conditions of stability at a plane joint are :

- I. That the portion of structure above the joint shall not overturn.
- II. That the maximum intensity of pressure at any point in the joint shall not exceed a certain limit known to be safe.
- III. That the portion of the structure above the joint shall not slide along the surface of the joint.

As the two first conditions are dependent on the position of the centre of pressure, these conditions may be stated as :

- I. The centre of pressure must fall within certain limiting positions on the surface of the joint.
- II. The angle between the direction of the resultant pressure and the normal to the joint must be less than the angle of friction.

It is well to note again with reference to Condition I., that if no *tensile* stress is permissible at any point in the surface of the joint the limiting distance of the centre of pressure from the centre of area is :

In a rectangular joint.  $\frac{1}{8}$ th the thickness of the joint.

In a circular joint.  $\frac{\text{Diameter}}{8}$ .

In a hollow circular joint of outside diameter  $D$ , and inside diameter  $D_1$ .  $\frac{D^2 + D_1^2}{8D}$ .

### 94. Consideration of the conditions of stability.

*Condition I.* Let Fig. 227 represent in section a portion of a pier or buttress,  $AB$  being the trace of one of the bed joints.



The polygon  $C_1C_2C_3$ , &c., formed by joining the successive centres of pressure by straight lines, is called the "*line of resistance*."

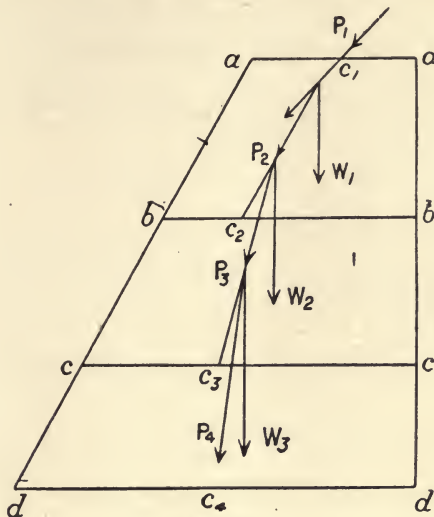


Fig. 228.

*Condition II.* The maximum intensity of pressure at any point of the joint shall not exceed a certain limit known to be safe.

When the position of the centre of pressure is determined the maximum intensity of pressure can be computed by equations (2) or (11).

The following data give the limiting intensities of pressure :

Rock, 8 to 12 tons per square foot.

Gravel and clay, 2 tons per square foot.

Loamy soil, 1 ton per square foot.

Good lime concrete, 3 to 4 tons per square foot.

*Condition III.* To satisfy this it is necessary that the tangential component of the resultant pressure shall not exceed the resistance of friction at the joint, which is the normal component multiplied by the coefficient of friction (or tangent of the angle of repose).

The following values may be taken for  $\phi$ , the angle of repose, and  $\mu = \tan \phi$  the coefficient of friction (Rankine) :

Surfaces	$\phi$	$\mu = \tan \phi$
Dry masonry and brickwork ...	31 to 35	·6 to ·7
Masonry and brick with wet mortar	25½	0·47
Masonry with damp mortar ...	36½	0·74
Masonry on dry clay... ..	27	0·51
Masonry on moist clay ... ..	18½	0·33

*Examples.*

1. A masonry pillar 4 feet diameter is built of masonry weighing 140 lbs. per cubic foot. It is subjected to a wind pressure whose normal intensity is 40 lbs. per square foot. Determine the greatest safe height of the pillar, assuming that, owing to the convexity of the pillar, the effective wind pressure per square foot is half the normal intensity on a plane section through the axis of pillar.

Let  $h$  = height of pillar.

„  $d$  = diameter = 4 feet.

„  $w$  = 140 lbs. per cubic foot.

Effective wind pressure =  $\frac{40}{2} = 20$  lbs. per square foot on a section through the axis of pillar.

Weight of pillar

$$= W = \frac{\pi d^2}{4} h w = \frac{22}{7} \times \frac{4^2}{4} \times 140 \times h = 1760h \text{ lbs.}$$

Total effective wind pressure =  $20 \times h \times 4$

$$= 80h.$$

Taking moments round the limiting position of the centre of pressure,

$$80h \times \frac{h}{2} = W \frac{d}{8} = 1760h \times \frac{1}{2} = 880h,$$

$$\therefore h = 22 \text{ feet.}$$

2. A pier of masonry weighing 112 lbs. per cubic foot, 20 feet high, and 6 feet square on plan, is subjected to a horizontal pressure of 4 tons applied to one face at a height of 12 feet above the footings. Investigate the stability of the joint at the top of the footings and calculate the greatest and least intensities of pressure (Fig. 229).

$P = 4$  tons.

Weight of pier

$$= W = 6 \times 6 \times 20 \times \frac{1}{20} = 36 \text{ tons.}$$

Then from triangle of forces  $CEO$ ,

$$\frac{x_0}{12} = \frac{4}{36}.$$

Therefore

$$x_0 = \frac{4 \times 8}{36} = 1\frac{1}{3} \text{ feet.}$$

But the limiting value of  $x_0$  is

$$\frac{1}{6} \times 6 = 1 \text{ foot.}$$

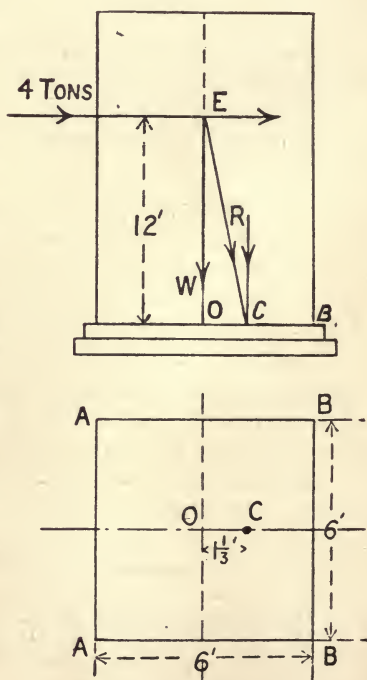


Fig. 229.

Therefore the condition that there shall be no tensile stress is *not* satisfied.

The maximum intensity of normal pressure is at the edge *B*.

The total normal component of  $R = W$ .

The maximum intensity of pressure

$$\begin{aligned} &= \frac{W}{S} + \frac{Wx_0x_1}{I} \\ &= \frac{36}{36} + \frac{36 \times 1\frac{1}{3} \times 3}{36 \times 3} \\ &= 2\frac{1}{3} \text{ tons per square foot.} \end{aligned}$$

3. A wall of thickness  $t$  retains water which is level with the top of the wall. Determine the height to which the wall can be built to satisfy the condition of the resultant falling within the middle third.

Let  $w$  = weight of water per cubic foot =  $62\frac{1}{2}$  lbs.

„  $w_1$  = weight of masonry per cubic foot.

If  $\rho$  = specific gravity of the masonry, then  $w_1 = \rho w$ .

$h$  = required height.

The total water pressure is  $\frac{wh^2}{2}$ .

The weight of masonry is  $w_1ht$ .

Take moments about the limiting position of centre of pressure

$$x_0 = \frac{t}{6}.$$

$$\text{Then } \frac{wh^2}{2} \cdot \frac{h}{3} = w_1 \frac{ht^2}{6} = w\rho \frac{ht^2}{6}.$$

$$\text{Therefore } h = t\sqrt{\rho}, \text{ or } t = \frac{h}{\sqrt{\rho}}.$$

4. Masonry wall with vertical face subjected to water pressure (Fig. 230).

Let  $t_1$  and  $t_2$  = thickness of wall at base and top respectively.

„  $w$  = weight of cubic foot of water.

„  $w_1$  = weight of cubic foot of masonry.

„  $\rho$  = specific gravity of the masonry.

Then  $w_1 = \rho w$ .

Assuming the water to be level with top of wall, find the height

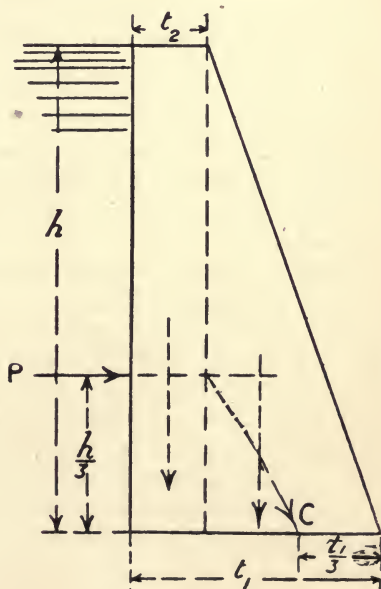


Fig. 230.

$h$  of the wall so that the resultant pressure will act at the outer middle third of the base.

*Note.* If the wall is *trapezoidal* or *triangular* in vertical section, and the resultant pressure acts within the middle third of the *base*, the same condition will be fulfilled for every other joint.

Consider a strip of wall 1 foot long. Divide up the section into a rectangle of area  $t_2 h$ , and a triangle of area  $\frac{t_1 - t_2}{2} h$ . The resultant water pressure is  $\frac{wh^2}{2}$  acting at a height  $\frac{h}{3}$  above the base. Taking moments about  $C$ , the outer middle third of  $t_1$ , we get

$$\begin{aligned}\frac{wh^3}{6} &= w_1 \left\{ t_2 h \left( \frac{2}{3} t_1 - \frac{t_2}{2} \right) + \frac{t_1 - t_2}{2} h \left[ \frac{2}{3} t_1 - t_2 - \frac{1}{3} (t_1 - t_2) \right] \right\} \\ &= \frac{w_1 h (t_1^2 + t_1 t_2 + t_2^2)}{6} = \frac{w \rho h}{6} (t_1^2 + t_1 t_2 + t_2^2).\end{aligned}$$

Therefore  $h^2 = \rho (t_1^2 + t_1 t_2 + t_2^2).$

The "line of resistance" in this case is a hyperbola. If the wall is rectangular in vertical section the "line of resistance" is a parabola.

If  $t_2 = 0$ , so that the vertical section of wall is a triangle of base  $t_1$ , and height  $h$ , we get

$$h^2 = \rho t_1^2.$$

Therefore  $h = t_1 \sqrt{\rho}$ , or  $t_1 = \frac{h}{\sqrt{\rho}}.$

The "line of resistance" is a straight line.

5. To find the height to which a dam of triangular section may be built consistent with the conditions of stability:

(a) That the resultant pressure, in its limiting position, shall cut the joint at  $\frac{2}{3}$  rds of its thickness from the inner face.

(b) That a given limiting intensity of pressure shall not be exceeded.

Assume the water level with the top of the dam (Fig. 231).

Let  $h$  = height of dam.

„  $t$  = thickness at base.

„  $W$  = total weight of masonry.

„  $\rho$  = specific gravity of masonry.

„  $R$  = resultant pressure on base.

„  $w$  = weight of one cubic foot of masonry.

„  $f$  = limiting intensity of pressure.

It was shown in the latter part of Example 4 that in the case of a triangular section in which condition (a) is fulfilled, that

$$t = \frac{h}{\sqrt{\rho}}.$$

The maximum intensity of pressure is on a plane at right angles to  $R$ , and if this intensity is  $f$  tons per square foot, then the intensity of pressure on the horizontal base  $AB$  is  $f \cos \theta$ .

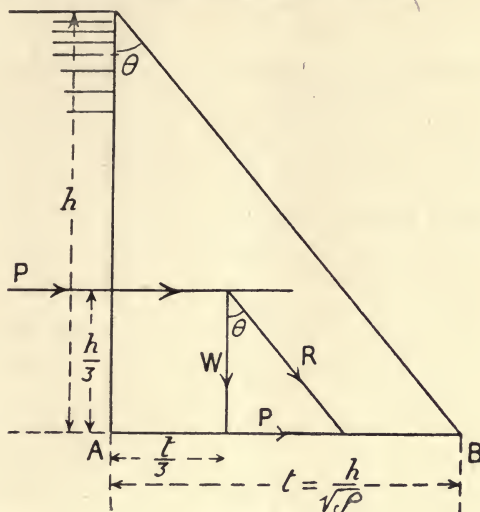


Fig. 231.

Therefore average intensity of pressure on  $AB = \frac{f \cos \theta}{2}$ .

But the average intensity of pressure on  $AB = \frac{R}{AB} = \frac{\frac{W}{\cos \theta}}{h \times \frac{1}{\sqrt{\rho}}}$ .

Therefore 
$$\frac{f \cos \theta}{2} = \frac{\frac{W}{\cos \theta}}{h \times \frac{1}{\sqrt{\rho}}},$$

or 
$$\frac{f \cos^2 \theta}{2} = \frac{W \sqrt{\rho}}{h} = \frac{h \rho w}{2}.$$

Since 
$$W = \frac{h^2}{2} \sqrt{\rho w},$$

but, 
$$\cos \theta = \frac{h}{\sqrt{t^2 + h^2}} = \sqrt{\frac{\rho}{1 + \rho}}.$$

Therefore, 
$$\frac{\rho}{1 + \rho} = \frac{h \rho w}{2}.$$

Hence 
$$f = h w (1 + \rho); \text{ or } h = \frac{f}{w (1 + \rho)}.$$

6. A cylindrical chimney is 10 feet outside diameter and 8 feet inside diameter. Find to what height such a chimney may be built, assuming that the brickwork weighs 112 lbs. per cubic foot and that the normal intensity of horizontal wind pressure is 40 lbs. per square foot (Fig. 232).

In the case of a cylindrical surface it is usual to assume that the effective intensity of the wind pressure is half the normal intensity, that is, half the intensity on a plane surface normal to the wind's direction.

Let  $H$  = height of chimney above its base.

„  $D$  = outside diameter of chimney.

„  $D_1$  = inside diameter of chimney.

„  $P$  = resultant wind pressure.

„  $f$  = effective intensity of the wind pressure.

„  $w$  = weight of masonry per cubic foot.

$P$  acts at a height of  $\frac{H}{2}$  above the base.

The greatest height  $H$  is found by equating the moment of the resultant wind pressure  $P$ , about the limiting position  $C$  of the centre of pressure of the base joint, to the moment of the weight of chimney about the same point.

The moment of  $P$  about  $C$  is

$$P \cdot \frac{H}{2} = fDH \cdot \frac{H}{2} = \frac{1}{2} fDH^2 \dots\dots\dots(1).$$

The weight of chimney is  $w \frac{\pi}{4} (D^2 - D_1^2) H$ .

The limiting distance of  $C$  from centre of area  $O$  is, for a hollow cylinder,

$$\frac{D^2 + D_1^2}{8D}.$$

Therefore the moment of the weight about  $C$  is

$$\begin{aligned} & wH \frac{\pi}{4} (D^2 - D_1^2) - \left( \frac{D^2 + D_1^2}{8D} \right) \\ &= \frac{\pi}{32} \frac{wH}{D} (D^2 - D_1^2) (D^2 + D_1^2) \dots\dots\dots(2). \end{aligned}$$

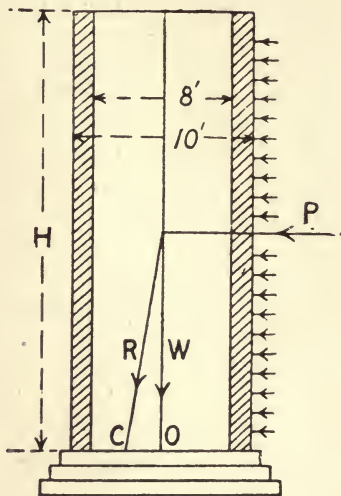


Fig. 232.

Equating (1) and (2) we get

$$\frac{1}{2} f D H^2 = \frac{\pi}{32} \frac{w H}{D} (D^2 - D_1^2) (D^2 + D_1^2).$$

Therefore limiting height is

$$H = \frac{\pi}{16} \frac{w}{f D^2} (D^2 - D_1^2) (D^2 + D_1^2) \dots\dots\dots(3).$$

Substituting the numerical values,

$$f = \frac{4.0}{2} = 20 \text{ lbs. per sq. ft.}, \quad w = 112 \text{ lbs. per sq. ft.}, \quad \pi = \frac{22}{7},$$

$$H = \frac{22}{7 \times 16} \frac{112}{20 \times 100} (100 - 64) (100 + 64) \\ = 65 \text{ feet.}$$

## 96. Earth pressure on retaining walls.

Let Fig. 233 represent a mass of earth supported by a retaining wall. If we imagine the wall removed, a wedge of earth  $ABC$  will separate from the general mass.

Let  $BC$  be the line along which rupture takes place inclined at an angle  $\theta$  to the vertical.

It is assumed that the pressure on the wall is the maximum pressure due to this wedge-shaped mass of earth; hence we require to find the value of  $\theta$  that will make the pressure on the wall a maximum.

Consider the wedge of earth  $ABC$ , the forces acting on it are:

(1) The weight  $W$  of the wedge acting at its centre of gravity.

(2) The reaction  $P$  of the plane  $AB$  acting at  $\frac{2}{3}AB$  from  $A$ .

(3) The friction along  $AB = \mu P$ ;  $\mu$  being the coefficient of friction.

(4) The reaction  $R$  of the plane  $BC$ .

(5) The friction along  $AC = \mu R$ . Assume that the coefficient of friction is the same for earth on earth as for earth on masonry. This although not quite true is sufficiently accurate for practical purposes, and simplifies the formulæ.

The forces in Fig. 233 represent these forces acting on the wedge of earth.

Resolving horizontally,

$$P + \mu R \sin \theta = R \cos \theta \dots\dots\dots(1).$$

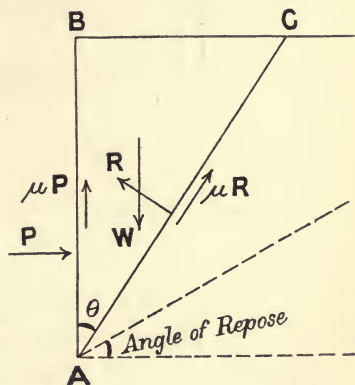


Fig. 233.

Resolving vertically,

$$\mu P + \mu R \cos \theta + R \sin \theta = W \dots\dots\dots(2).$$

Let  $\rho$  = specific gravity of earth ; and take the weight of 1 cubic foot of water =  $\frac{1}{8}$  ton. Then

$$W = \frac{h \times h \tan \theta}{2} \times \rho \times \frac{1}{8} \text{ ton} \\ = \frac{\rho h^2 \tan \theta}{72}.$$

Substituting this value of  $W$  in (2),

$$\mu P + \mu R \cos \theta + R \sin \theta = \frac{\rho h^2 \tan \theta}{72} \dots\dots\dots(3).$$

From (1),  $R (\cos \theta - \mu \sin \theta) = P.$

Therefore  $R = \frac{P}{\cos \theta - \mu \sin \theta}.$

Substituting this value of  $R$  in (3),

$$\mu P + \frac{\mu P \cos \theta}{\cos \theta - \mu \sin \theta} + \frac{P \sin \theta}{\cos \theta - \mu \sin \theta} = \frac{\rho h^2}{72} \tan \theta,$$

or  $\frac{P (2\mu \cos \theta - \mu^2 \sin \theta + \sin \theta)}{\cos \theta - \mu \sin \theta} = \frac{\rho h^2}{72} \cdot \frac{\sin \theta}{\cos \theta}.$

Therefore  $P = \frac{\rho h^2}{72} \cdot \frac{1 - \mu \tan \theta}{1 - \mu^2 + 2\mu \cot \theta} \dots\dots\dots(4).$

To find the value of  $\theta$ , for which  $P$  is a maximum, equate  $\frac{dP}{d\theta}$  to zero.

That is,

$$(1 - \mu^2 + 2\mu \cot \theta) (-\mu \sec^2 \theta) - (1 - \mu \tan \theta) (-2\mu \operatorname{cosec}^2 \theta) = 0,$$

or  $-(1 - \mu^2) \sin^2 \theta - 2\mu \sin \theta \cos \theta + 2 \cos^2 \theta - 2\mu \sin \theta \cos \theta = 0.$

Therefore  $(1 - \mu^2) \tan^2 \theta + 4\mu \tan \theta - 2 = 0.$

Solving, we get for a maximum

$$\tan \theta = \frac{-2\mu + \sqrt{2(\mu^2 + 1)}}{1 - \mu^2} \dots\dots\dots(5).$$

Substituting this value of  $\tan \theta$  in equation (4),

$$\text{Max. } P = \frac{\rho h^2}{72} \cdot \frac{3\mu^2 - 2\mu \sqrt{2(\mu^2 + 1)} + 1}{(1 - \mu^2)^2}.$$

**97. To find the resultant pressure on the base of a retaining wall, in magnitude, direction, and position.**

Let  $t$  be the thickness of base (Fig. 234).

„  $h$  be the height of wall.

„  $d$  be the distance of the point where the resultant cuts the base from the outer edge of base.

Let  $P$  be the maximum pressure of the earth on wall acting at a height  $\frac{h}{3}$  above base.

„  $W_1$  be the weight of the wall.

„  $R$  be the resultant pressure on base of wall.

„  $\alpha$  be the angle which the direction of resultant makes with horizontal.

„  $b$  be the distance of line of action of  $W_1$  from outer edge of base.

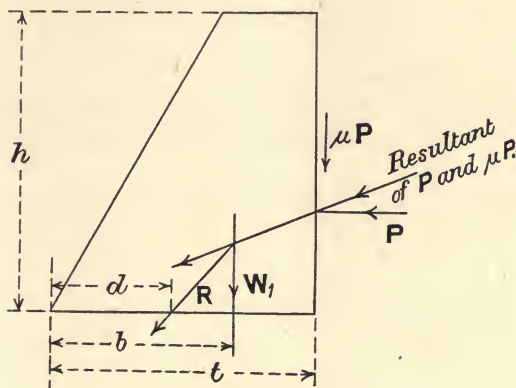


Fig. 234.

Then resolving horizontally and vertically,

$$R \cos \alpha = P,$$

$$R \sin \alpha = W_1 + \mu P.$$

Therefore

$$R^2 = P^2 + (W_1 + \mu P)^2 \dots \dots \dots (6),$$

and

$$\tan \alpha = \frac{W_1 + \mu P}{P} \dots \dots \dots (7).$$

Taking moments about outer edge of base,

$$Rd \sin \alpha = W_1 b + \mu P t - P \frac{h}{3}.$$

Therefore

$$d = \frac{W_1 b + \mu P t - P \frac{h}{3}}{W_1 + \mu P} \dots \dots \dots (8).$$

Equations (6), (7), (8), give the magnitude, direction and position of the resultant pressure.

## CHAPTER X.

### COLUMNS AND STRUTS.

#### 98. Short columns.

If a load  $P$  acts along the axis of a short column, the ratio of whose length to diameter is small, usually not greater than 3 to 1, the column will fail by direct crushing.

The relation between the load and the stress produced is

$$f = \frac{P}{S},$$

where  $f$  = intensity of compressive stress,

$P$  = total load,

$S$  = area of cross section.

COLUMNS OF MEDIUM LENGTH fail partly by crushing and partly by bending.

VERY LONG COLUMNS fail wholly by bending.

#### 99. Rankine's and Gordon's formulæ for columns of medium length.

Let the column be hinged or free at both ends (Fig. 235).

Let  $OA$  be the primitively straight axis of the column; take this as axis of  $x$ , and the extremity  $O$  for origin.

„  $l$  = length of the column, the deformed axis after bending being represented by the curve  $OBA$ .

„  $x$  and  $u$  = the coordinates of the centre of area of a normal cross section,  $u$  being the deflection at abscissa  $x$ .

„  $u_1$  = maximum deflection.

„  $P$  = load.

„  $S$  = area of cross section.

„  $f_1$  = maximum intensity of stress.

Then, using the notation of previous chapter, if

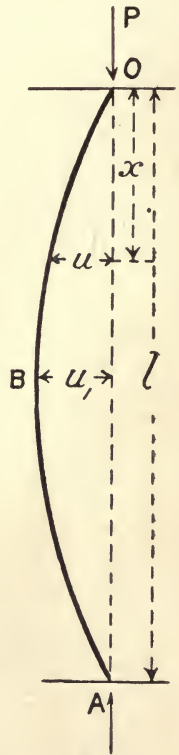


Fig. 235.

$f_0$  be the uniform compressive stress on a cross section of the column (Fig. 236),

$$f_0 = \frac{P}{S} \dots \dots \dots (1).$$

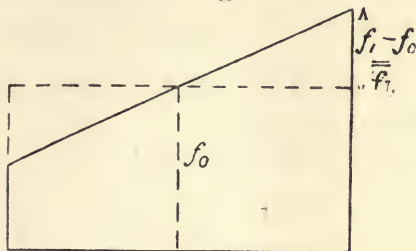


Fig. 236.

Let  $f_b = f_1 - f_0$  be the maximum intensity of stress due to bending.

We know that  $f_b = \frac{My}{I}$ .

But Max.  $M = Pu_1$ , where  $u_1$  is the deflection of the middle of the column from vertical.

Therefore  $f_b = \frac{Pu_1 \cdot y}{I} \dots \dots \dots (2).$

To find  $u_1$  let  $R$  be the radius of curvature of the bent axis of column (Fig. 237), then

$$\left(\frac{l}{2}\right)^2 = u_1 (2R - u_1) = 2Ru_1 \text{ (app.)},$$

$$\therefore u_1 = \frac{l^2}{8R} \dots \dots \dots (3),$$

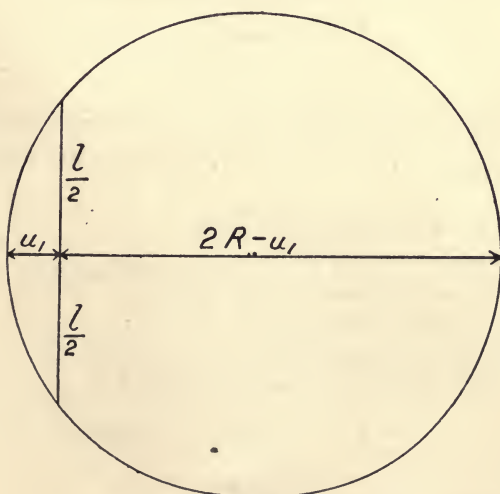


Fig. 237.

but  $\frac{f_b}{y} = \frac{E}{R}$ . Therefore from (3)

$$u_1 = \frac{f_b}{8E} \frac{l^2}{y} = a \frac{l^2}{y}, \text{ where } a \text{ is a constant} = \frac{f_b}{8E}.$$

Substituting in equation (2) this value of  $u_1$ , we get

$$f_b = \frac{P}{I} a l^2 = \frac{P}{S} a \frac{l^2}{k^2} = f_0 a \frac{l^2}{k^2},$$

where  $k$  is the radius of gyration.

Then the maximum intensity of stress,

$$f_1 = f_0 + f_b = f_0 \left( 1 + a \frac{l^2}{k^2} \right) = \frac{P}{S} \left( 1 + a \frac{l^2}{k^2} \right).$$

If  $f_1$  is the maximum stress intensity allowable, which must not be greater than the elastic limit stress in compression of the material, then

$$P = \text{breaking load} = \frac{f_1 S}{1 + a \frac{l^2}{k^2}} \dots\dots\dots (4),$$

where  $k$  is the radius of gyration of section with respect to the axis about which the resistance to bending is least, namely, the axis about which  $I$  is least.

Equation (4) is *Rankine's formula*, and the constant  $a$  depends on the material.

The *steady* working load should not be greater than  $nP$ , where  $n = \frac{1}{4}$ th or  $\frac{1}{5}$ th for wrought iron and steel,  $\frac{1}{3}$ th for cast iron, and  $\frac{1}{10}$ th for wood. For *live loads* these values of  $n$  must be halved.

*Gordon's formula*, which is similar to Rankine's, is

$$P = \frac{f_1 S}{1 + c \frac{l^2}{b^3}} \dots\dots\dots (5),$$

where  $b$ , the least breadth of the section, is used instead of  $k$ , the least radius of gyration. The constant  $c$  depends both on the material and on the type of section.

### 100. Proof of Gordon's formula.

Taking the same notation as in the proof of Rankine's formula,

Let  $b$  be the *least* dimension of the section.

„  $d$  be the greater dimension of the section.

„  $S = db$  = area of cross section.

The maximum bending moment,  $M = Pu_1$ ;

but  $u_1 \propto \frac{l^2}{b}$ . (Deflection of Beams.)

Therefore,  $M \propto \frac{Pl^2}{b}$ .

Again,  $f_b \propto \frac{M}{db^2} \propto \frac{Pl^2}{db^3} \propto \frac{Pl^2}{Sb^2}$   
 $= cf_0 \frac{l^2}{b^2}$ , where  $c$  is a constant.

Now, maximum intensity of stress,

$$f_1 = f_0 + f_b = f_0 \left( 1 + c \frac{l^2}{b^2} \right) = \frac{P}{S} \left( 1 + c \frac{l^2}{b^2} \right).$$

Therefore,

$$P = \frac{f_1 S}{1 + c \frac{l^2}{b^2}}.$$

### 101. Fixed and hinged ends.

In Rankine's and Gordon's formulæ  $l$  is the length of the column between the supports. If both ends of the column are *fixed* (Fig. 239)

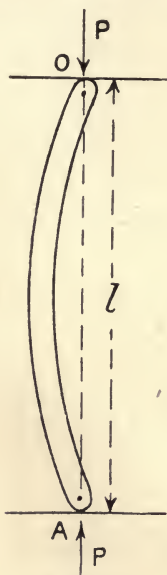


Fig. 238.

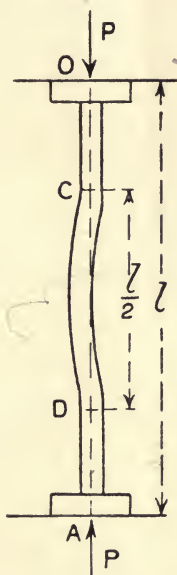


Fig. 239.



Fig. 240.

the load which it will carry before bending is the same as for a strut of half the length hinged at the ends, and we must substitute  $\frac{l}{2}$  for  $l$ .

If one end of the column is fixed and the other end rounded or free (Fig. 240) we must substitute  $\frac{2}{3}l$  for  $l$ . There are thus three cases to consider. Rankine's formulæ for the three cases are :

*Both ends rounded or free,*

$$P = \frac{f_1 S}{1 + a \frac{l^2}{k^2}} \dots \dots \dots (6).$$

*Both ends fixed,* 
$$P = \frac{f_1 S}{1 + \frac{a}{4} \frac{l^2}{k^2}} \dots \dots \dots (7).$$

*One end fixed and the other rounded or free,*

$$P = \frac{f_1 S}{1 + \frac{4}{9} a \frac{l^2}{k^2}} \dots \dots \dots (8).$$


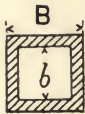
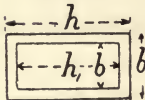
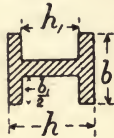
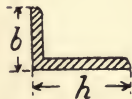
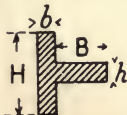
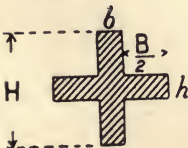
The following values found from experiment may be taken for the constants  $f_1$  and  $a$  in Rankine's formulæ (6), (7), (8) :

Material	Tons per Square Inch	$a$
Cast iron ... ..	35	$\frac{1}{1800}$
Wrought iron... ..	16	$\frac{1}{9000}$
Mild steel ... ..	21	$\frac{1}{7500}$
Hard steel ... ..	30	$\frac{1}{5000}$

The following table gives the values of  $c$  and  $f_1$  in Gordon's formula, equation (5), found from experiments :

Material	Section	Values of $c$			$f_1$
		Ends round or pivoted	Ends fixed	One end fixed, the other end pivoted	
Wrought iron	L, T, channel	$\frac{4}{900}$	$\frac{1}{900}$	$\frac{1}{300}$	19
	I and hollow square }				
	Hollow round				
	Solid round				
Cast iron	Solid rectangular	$\frac{7}{100}$	$\frac{3}{3000}$	$\frac{1}{1700}$	16
	Hollow round	$\frac{2}{100}$	$\frac{8}{800}$	$\frac{4}{400}$	35
	Solid round	$\frac{1}{100}$	$\frac{4}{400}$	$\frac{2}{230}$	35
	Solid rectangular	$\frac{6}{200}$	$\frac{2}{1480}$	$\frac{1}{1350}$	30
Mild steel	Solid round	$\frac{3}{350}$	$\frac{1}{1400}$	$\frac{7}{780}$	30
	Hollow round	$\frac{6}{25}$	$\frac{2}{500}$	$\frac{1}{1400}$	30

*Values of  $k^2$  for different Cross sections.*

Cross section	Least $k^2$
 Square of side $b$ , or rectangle with smallest side $b$	$\frac{b^2}{12}$
 Hollow square ... ..	$\frac{B^2 + b^2}{12}$ or $\frac{B^2}{6}$ if thin
 Hollow rectangle... ..	$\frac{bb^3 - b_1b_1^3}{12 (bh - b_1h_1)}$
 H section, equal flanges ...	Do.
 Angle-iron, smallest side $b$ ...	$\frac{1}{12} \frac{b^2 h^2}{b^2 + h^2}$ app.
 Tee-iron, smallest side $H$ ...	$\frac{bH^3 + Bh^3}{12 (Bh + bH)}$
 Cruciform do. ... ..	Do.
Angle, tee, or cruciform section with equal sides, each equal to $b$	$\frac{b^2}{24}$
Solid circle, diameter $d$ ... ..	$\frac{d^2}{16}$
Hollow circle, external diameter $D$ , internal diameter $d$	$\frac{D^2 + d^2}{16}$
Thin hollow circle, external diameter $D$ ... ..	$\frac{D^2}{8}$ app.

### 102. Very long columns. Euler's formula.

Euler's formula, which is applicable only to columns where the ratio of length to diameter is very great, is

$$P = \frac{\pi^2 EI}{l^2},$$

where  $P$  is the limiting load which the strut can support.

$E$  is the modulus of elasticity.

$I$  the least moment of inertia of the cross section about an axis through the centre of area of the section.

$l$  the length of the strut.

This equation is founded on the following assumptions, which are never really fulfilled in practice, and is consequently that for an ideal column :

- (1) *The column is originally straight and of uniform section.*
- (2) *The line of action of the load coincides initially with the axis of the column.*
- (3) *Material of the column homogeneous.*

### 103. Proof of Euler's formula.

Suppose the column hinged or rounded at the ends. Let  $OBA$  (Fig. 241) be the bent axis of the column. Take  $O$  as origin and the vertical line  $OA$  as the axis of  $x$ . Let  $u$  be the deflection at any point  $C$ .

Then  $M = -Pu = EI \frac{d^2 u}{dx^2}$ . (See Chapter on Deflection.)

The negative sign is used, because, if deflection is positive, the centre of curvature lies on the negative side.

$$\frac{d^2 u}{dx^2} = -\frac{Pu}{EI},$$

$$\frac{du}{dx} \frac{d^2 u}{dx^2} = -\frac{Pu}{EI} \frac{du}{dx}.$$

Integrating,  $\left(\frac{du}{dx}\right)^2 = -\frac{P}{EI} (u^2 + c);$

when  $\frac{du}{dx} = 0$ ,  $u = \delta$  the maximum deflection, hence  $c = -\delta^2$ .

Therefore  $\left(\frac{du}{dx}\right)^2 = \frac{P}{EI} (\delta^2 - u^2),$

or  $dx = \sqrt{\frac{EI}{P}} \frac{du}{\sqrt{\delta^2 - u^2}}.$

Integrating again,  $x = \sqrt{\frac{EI}{P}} \sin^{-1} \frac{u}{\delta} + c_1;$

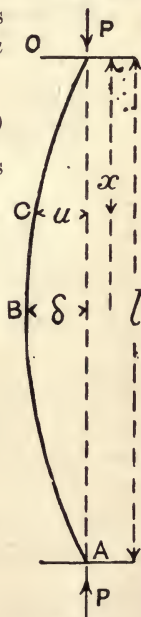


Fig. 241.

when  $x = 0$ ,  $u = 0$ , therefore  $c_1 = 0$ ,

therefore 
$$u = \delta \sin \left( x \sqrt{\frac{P}{EI}} \right),$$

which is the equation of the elastic curve.

Again, when  $x = \frac{l}{2}$ ,  $u = \delta$ ;

therefore 
$$\sin \frac{l}{2} \sqrt{\frac{P}{EI}} = 1,$$

or 
$$\frac{l}{2} \sqrt{\frac{P}{EI}} = \frac{\pi}{2}, \text{ or } \frac{3\pi}{2}, \text{ or } \frac{5\pi}{2}, \text{ \&c.}$$

The *least* value of  $P$ , and hence the *minimum* thrust which will bend the column, is given by

$$\frac{l}{2} \sqrt{\frac{P}{EI}} = \frac{\pi}{2},$$

or

$$P = \frac{\pi^2 EI}{l^2} \dots \dots \dots (9).$$

But  $P = pS$ , and  $I = Sk^2$ ;

hence

$$p = \frac{\pi^2 E}{\left(\frac{l}{k}\right)^2}.$$

#### 104. Fixed ends.

If the strut is fixed at both ends, the load which it will stand before yielding is the same as for a strut of half the length hinged at the ends (Figs. 238 and 239).

*If the ends are fixed,*

$$P = \frac{\pi^2 EI}{\left(\frac{l}{k}\right)^2} = \frac{4\pi^2 EI}{l^2} \dots \dots \dots (10).$$

*If one end is fixed and the other hinged,*

$$P = \frac{\pi^2 EI}{\left(\frac{2}{3}l\right)^2} = \frac{9\pi^2 EI}{4l^2} \dots \dots \dots (11).$$

Euler's equation is for very long struts loaded under ideal conditions of initial straightness and perfectly axial load combined with a perfectly homogeneous material, and requires modification to render it applicable to practical cases.

For very short columns, if  $f$  is the crushing strength of the material and  $S$  the area of cross section,

Breaking load  $= fS$ .

For very long columns, according to Euler's theory,

$$\text{Breaking load} = \frac{\pi^2 EI}{l^2}.$$

Then the formula

$$P = \text{breaking load} = \frac{fS}{1 + fS \frac{l^2}{\pi^2 EI}} \dots \dots \dots (12)$$

may be taken as true for columns of all lengths, because in this formula, if  $l$  is *small*, the denominator is 1 (app.) and  $P = fS$ .

When  $l$  is *great*, we can neglect 1 in the denominator, and

$$P = \frac{\pi^2 EI}{l^2}.$$

Let  $I = Sk^2$ , where  $k$  is the least radius of gyration of the section, then we get from (12)

$$P = \frac{fS}{1 + a \frac{l^2}{k^2}} \dots \dots \dots (13),$$

where  $a = \frac{f}{\pi^2 E}$ ; but if  $a$  is calculated from Euler's formula, we get values which make the strut too strong, because in practice perfect straightness, symmetrical loading, and symmetry of elasticity do not exist; hence the formula is treated as empirical, and the constants  $f$  and  $a$  are determined from experimental results.

### 105. Johnson's parabolic formula.

In Euler's formula the buckling stress is

$$p = \frac{P}{S} = \frac{\pi^2 E}{\left(\frac{l}{k}\right)^2}.$$

In Rankine's formula

$$p = \frac{P}{S} = \frac{f}{1 + a \left(\frac{l}{k}\right)^2},$$

where  $f$  is the "elastic limit" stress in compression.

Professor Johnson has deduced the formula

$$p = \frac{P}{S} = f - b \left(\frac{l}{k}\right)^2,$$

where  $f$  is the "elastic limit" stress in compression of the material, and  $b$  is a constant whose value is

$$b = \frac{f^2}{4\pi^2 E}.$$

If a curve is plotted as in Fig. 242 representing Euler's formula when applied to wrought-iron columns, with ratios of  $l$  to  $k$  as abscissæ, and buckling stresses as ordinates, then Johnson's formula

$$p = f - \frac{f^2}{4\pi^2 E} \left(\frac{l}{k}\right)^2 = f - b \left(\frac{l}{k}\right)^2$$

is the equation to a curve parabolic in form, tangential to Euler's curve, where  $\frac{l}{k} = 150$ , and with its apex at the elastic limit of the metal.

Professor Johnson gives the following values as deduced from the Watertown experiments :

For wrought iron  $f$  is taken as 34000 lbs. per square inch.

For mild steel  $f$  is taken as 42000 lbs. per square inch.

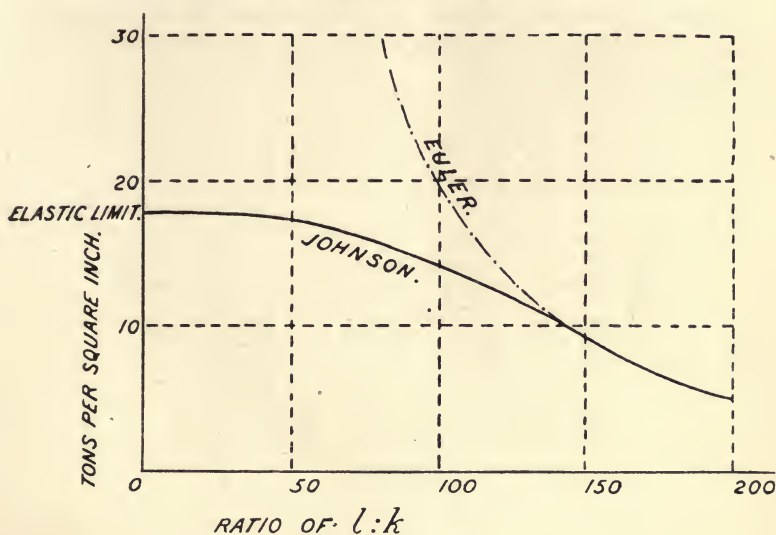


Fig. 242.

*Wrought-iron columns, pin ends,*

$$\frac{l}{k} < 170 ; p = 34000 - 0.67 \left( \frac{l}{k} \right)^2.$$

*Wrought-iron columns, flat ends,*

$$\frac{l}{k} < 210 ; p = 34000 - 0.43 \left( \frac{l}{k} \right)^2.$$

*Mild-steel columns, pin ends,*

$$\frac{l}{k} < 150 ; p = 42000 - 0.97 \left( \frac{l}{k} \right)^2.$$

*Mild-steel columns, flat ends,*

$$\frac{l}{k} < 190 ; p = 42000 - 0.62 \left( \frac{l}{k} \right)^2.$$

*Cast-iron columns, round ends,*

$$\frac{l}{k} < 70 ; p = 60000 - \frac{25}{4} \left( \frac{l}{k} \right)^2.$$

*Cast-iron columns, flat ends,*

$$\frac{l}{k} < 120; \quad p = 60000 - \frac{9}{4} \left( \frac{l}{k} \right)^2.$$

The working stress must not exceed  $\frac{p}{n}$ , where  $n=4$  to 5 for wrought-iron and steel, and 6 for cast-iron.

*Example.*

*Find by Rankine's formula the working load for a wrought-iron column 25 feet long, firmly fixed at the ends, and the section of which is given in sketch (Fig. 243).*

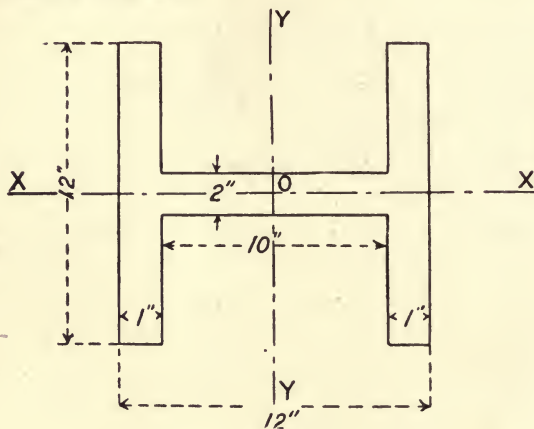


Fig. 243.

Assume  $f=36000$  lbs. per square inch, and a factor of safety = 5.

In this case the *least* moment of inertia is evidently about the axis  $XOX$ , and

$$I_{XOX} = \frac{2 \times 1 \times 12^3 + 10 \times 2^3}{12} = 294.67.$$

Area of section,  $S = 2 \times 12 \times 1 + 2 \times 10 = 43$  square inches.

$$\text{Therefore} \quad k^2 = \frac{I}{S} = \frac{294.67}{44} = 6.7.$$

Also  $l = 25$  feet = 300 inches.

Substituting these values in Rankine's formula, we get the breaking

$$\begin{aligned} \text{load} \quad P &= \frac{36000 \times 44}{1 + \frac{1}{36000} \cdot \frac{300^2}}{1 + \frac{100}{268}} \\ &= 1153565 \text{ lbs.} = 515 \text{ tons.} \end{aligned}$$

With the given factor of safety 5, the working load will be

$$\frac{515}{5} \text{ tons} = 103 \text{ tons.}$$

The same by Johnson's parabolic formula.

$$\begin{aligned}
 P &= S \left[ 34000 - 0.67 \left( \frac{l}{k} \right)^2 \right] \\
 &= 44 \left[ 34000 - 0.67 \cdot \frac{90000}{6.7} \right] \\
 &= 1100000 \text{ lbs.} \\
 &= 500 \text{ tons.}
 \end{aligned}$$

Therefore working load  $= \frac{500}{5} = 100$  tons.

### EXERCISES ON CHAPTERS IX. AND X.

1. An upright post 14 inches  $\times$  11 inches supports a vertical load of 15 tons; the resultant acts in the centre line of the width (least dimension) of post, but at a distance of 4 inches from the centre of area. Determine the mean and maximum intensities of stress occurring on a normal cross section.

*Ans.* Mean intensity of stress  $= 0.097$  ton per sq. inch compression.

Max. " " "  $= 0.264$  " " " "

Min. " " "  $= -0.07$  " " " tension.

2. A T-iron consists of a web half an inch thick and 4 inches deep, with a flange half an inch thick and 2 inches broad. It is subject to tension, and the line of action of the tensile force acts through the centre of the depth instead of along the axis of the piece. Find in what ratio the T-iron is weakened.

*Ans.* 1.68 to 1.

3. A short vertical pillar of mild steel, 5 inches diameter, carries a load of 50 tons acting vertically at 1 inch from the axis. Determine the maximum intensities of stress on a cross section.

*Ans.* 6.62 tons per square inch compressive.

1.53 " " " tensile.

4. A short hollow circular column of cast iron, 6 inches external diameter and 1 inch thick, carries a load of 60 tons, the line of action of which is parallel to the vertical axis, but  $1\frac{1}{4}$  inches distant from it. Find the greatest intensities of stress on a normal cross section.

*Ans.* 8.23 tons per square inch compressive.

0.59 " " " tensile.

5. A vertical pier of brickwork of rectangular section, 5 feet by 6 feet, and 15 feet high above the footings, receives at the top an inclined downward thrust, whose vertical component, acting along the vertical axis of the pier, is 10 tons, and whose horizontal component at the top of the pier, acting parallel to the 6-feet width, is 3 tons. Calculate the position of the centre of pressure on the joint above footings, and determine the maximum intensity of compressive stress.

Assume the brickwork to weigh 112 lbs. per cubic foot.

*Ans.*  $x_0 = 1.38$  feet; 41.6 lbs. per square inch.

6. The total vertical pressure on a horizontal section of a wall is 60 tons per foot of length. The thickness of wall is 6 feet, and the centre of pressure is 6 inches from the centre of thickness of joint. Determine the intensity of stress at the opposite edges of the joint.

*Ans.* 15 tons per square foot ; 5 tons per square foot.

7. Calculate by Rankine's formula the safe load for a hollow cylindrical cast-iron column 10 feet long, 6 inches internal and 7 inches external diameter, (1) when fixed at both ends ; (2) when hinged at both ends.

Take  $f = 36$  tons per square inch, and a factor of safety = 6.

*Ans.* (1) 43.03 tons ; (2) 22.74 tons.

8. Find by the same formula the load which an angle-iron strut of section 3 inches by 3 inches by  $\frac{3}{8}$  inch and 12 feet long will carry (1) when fixed at both ends ; (2) when hinged at both ends. Assume the intensity of working stress to be 5 tons per square inch.

*Ans.* 3.6 tons ; 1.21 tons.

$$(k^2 = 0.299).$$

9. A wrought-iron tubular strut, in a roof truss, carries a compressive load of 6 tons acting along the axis ; the strut is 8 feet long,  $2\frac{1}{2}$  inches external diameter and  $\frac{3}{8}$  inch thick. Find the maximum intensity of stress to which a cross section is liable : (1) strut hinged at both ends ; (2) fixed at one end and hinged at the other.

*Ans.* 6.6 tons per square inch.

4.27 tons per square inch.

10. Find by Gordon's formula the breaking weight of a wrought-iron strut of angle-iron section  $3'' \times 3'' \times \frac{3}{8}''$ , its length being 5 feet, and ends fixed.

Assume  $f = 19$  tons,  $\alpha = \frac{1}{900}$ .

*Ans.* 21.2 tons.

11. Find by Rankine's formula the breaking weight of a cast-iron column 30 feet long, 12 inches external diameter and  $\frac{1}{2}$  inch thick, both ends being hinged.

*Ans.* 125 tons.

12. A strut 10 feet long is made up of two tee-irons 6 inches by 3 inches by  $\frac{1}{2}$  inch, riveted back to back. Determine the working load by Gordon's formula :

(1) When the ends are fixed.

(2) When the ends are hinged.

Assume  $f = 19$  tons per square inch ;  $c = \frac{1}{900}$ , and take a factor of safety of 5.

*Ans.* (1) 22.3 tons ; (2) 11.6 tons.

13. Find by Gordon's formula the working load for a cast-iron pillar 8 inches external diameter,  $6\frac{1}{2}$  inches internal diameter, and 22 feet high, both ends being fixed. Take a factor of safety of 10, and assume  $f = 35$  tons per square inch;  $c = \frac{1}{800}$ .

Ans. 21.25 tons.

14. Fig. 244 represents a vertical section of a wall which has to resist the pressure of water on either side alternately, *i.e.*, not simultaneously on both sides. Assuming that there shall be no tension at any joint, and that the weight of a cubic foot of water is  $w$ , and of a cubic foot of masonry  $2w$ , find the thicknesses  $t_1$  and  $t_2$ . Take the water as level with the top of wall.

Ans.  $t_1 = 2.12$  feet;  $t_2 = 4.35$  feet.

15. A reservoir wall is vertical on the inner face; it is 10 feet thick at the base, 3 feet thick at the top, and 15 feet high. The water is liable to rise to the top of the wall.

(A) Determine the positions of the centre of pressure on the horizontal base joint: (1) when the reservoir is empty; (2) when the reservoir is full.

(B) Are the conditions of stability fulfilled?

(C) The reservoir being full, what are the intensities of pressure on the base at its inner and outer edges?

Assume the weight of wall 120 lbs. per cubic foot, and of water 62.4 lbs. per cubic foot.

Ans. (A) (1) 1.44 feet from centre of base towards inner face.

(2) 1.56 " " " " " " " " outer face.

(B) Yes.  $\left\{ \begin{array}{l} 1. \text{ Centre of pressure falls within the middle third.} \\ 2. \text{ Moment of weight 36352 foot lbs. exceeds moment of water pressure 35100 foot lbs.} \\ 3. \text{ Tangent of inclination of resultant with vertical} = \frac{7020}{11700} = 0.6. \end{array} \right.$

(C)  $f_1 = 2265$  lbs. per square foot.

$f_2 = 74.9$  " " " "

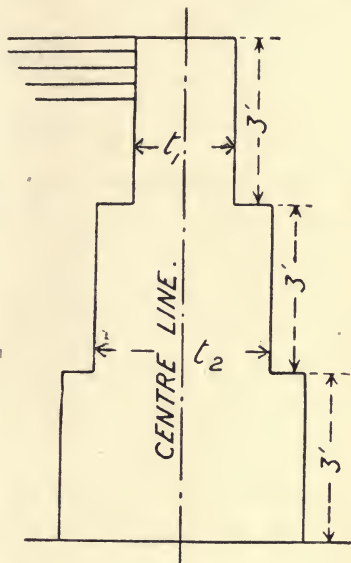


Fig. 244.

## CHAPTER XI.

### RIVETED JOINTS.

#### 106. Definitions. Lap and butt joints.

In a *lap joint* one plate overlaps the other, and they are connected by one or more rows of rivets.

In a *butt joint* the plates are kept in the same plane, and the joint is covered on one or both sides by a *cover plate*, and riveted to each. *by one or more rows of rivets.*

The lap joint is objectionable, owing to the straining forces on the two plates not being in the same line, thus forming a couple, which weakens the joint by bending (Fig. 247).

The butt joint is the one generally used, and is the more effective joint, owing to its symmetry and the absence of eccentric stresses.

*Single riveting* is when there is only one line of rivets in a lap joint, or one line on *each* side of the joint in a butt joint.

*Double riveting*, when there are two lines of rivets in the lap, or two lines on *each* side of the joint in a butt joint.

Fig. 245 shows a single-riveted lap joint; Fig. 246 a single-riveted butt joint; Figs. 248 to 251 show double-riveted lap and butt joints.

In *chain riveting* the rivets in the several rows are opposite to one another (Figs. 248 and 250).

In *zig-zag riveting* the rivets in one row alternate with the spaces in next row (Figs. 249 and 251).

• The *pitch* is the distance from centre to centre of the rivets in one row.

The *lap* is the distance at right angles to the joint, between the edges of two overlapping plates; or, in the case of a butt joint, the distance between the joint and the end of the cover plate.

A rivet is in *single shear* when shearing can take place only on *one* cross section of the rivet, as in lap joints and in butt joints with one cover plate (Figs. 245 and 246).

A rivet is in *double shear* when shearing can take place on two cross sections, as in butt joints with two cover plates.

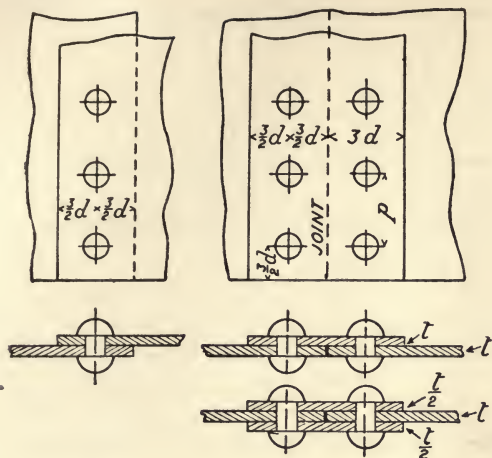


Fig. 245.

Fig. 246.



Fig. 247.

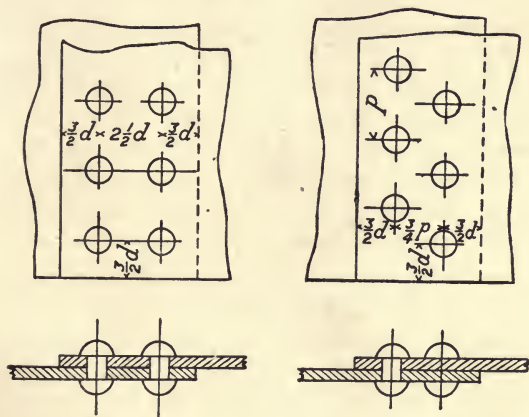


Fig. 248.

Fig. 249.

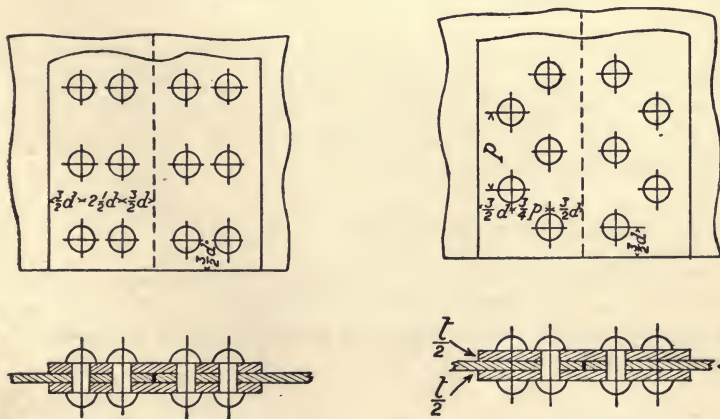


Fig. 250.

Fig. 251.

### 107. Rules to be observed in designing joints.

#### *Diameter of Rivets for given Plates.*

Let  $t$  = thickness of plate in inches.

„  $d$  = diameter of rivet in inches.

The following rule is sometimes used :  $d = 2t$  for plates under  $\frac{1}{2}$ " ;  
 $d = 1\frac{1}{2}t$  for plates of  $\frac{1}{2}$ " and over.

Professor Unwin gives the simple rule which should be adopted :

$$d = 1.2 \sqrt{t}.$$

In girder-work the rivets ought, if possible, to be of one size throughout, or at most two sizes. In structural iron-work of this class rivets  $\frac{3}{4}$ " and  $\frac{7}{8}$ " are most generally used. Field rivets, which have to be riveted up by hand when the girder is in position, should never exceed  $\frac{3}{4}$ " diameter, on account of the difficulty of driving tight rivets of larger size by hand.

*Minimum pitch.* The pitch of the rivets, as will be seen presently, is found by equating the shearing strength of the rivets to the tensile strength of the net area of the plate, but the distance between the edges of the rivet holes should *never be less than* the diameter of the rivet. This gives the *minimum* pitch =  $2d$ .

In boiler-work the pitch of the rivets is necessarily close, but in *girder-work* the pitch is practically *never less* than three diameters.

A maximum pitch of 6" should not be exceeded, as it is advisable to keep the plates close to prevent the entrance of water.

*The distance from the centre of rivet hole to the edge of a plate* should not be *less than*  $1\frac{1}{2}d$ . This leaves a clear diameter of rivet between the edge of hole and edge of plate. This minimum distance is, in practice, increased to  $1\frac{1}{2}d + \frac{1}{16}$ , and in girder-work is often  $2d$ . It should be noted that the diameter of the hole is usually  $\frac{1}{16}$  of an inch larger than the diameter of the rivet, to allow the latter to enter when hot.

The *grip* of a rivet—that is, the distance between its heads—is the thickness of the plates to be joined by it, plus  $\frac{1}{32}$  of an inch for each joint between the plates to allow for uneven surfaces, which prevents very close contact. The maximum grip of a rivet should not exceed *four times the diameter of the rivet*.

### 108. Strength of riveted joints.

Take, for simplicity, the case of a single-riveted lap joint. Consider a strip of such a joint of width equal to the pitch (Fig. 252). As each rivet supports such a strip, the results obtained may be applied to the joint as a whole.

Let  $p$  = pitch of rivets.

„  $d$  = diameter of rivet.

„  $t$  = thickness of plates.

„  $l$  = distance from centre of rivet to edge of plate.

„  $f_t$  = tensile resistance of plates.

„  $f_s$  = shearing resistance of rivets.

„  $f_c$  = crushing resistance.

„  $T$  = resistance of a strip of the joint of width  $p$ .

Such a joint, if in tension, may fail in four ways :

1. The rivet may shear (Fig. 253). The area resisting shear

$$= \frac{\pi d^2}{4}.$$

The resistance to shear is

$$T = f_s \frac{\pi d^2}{4} \dots \dots \dots (1).$$

Fig. 252.

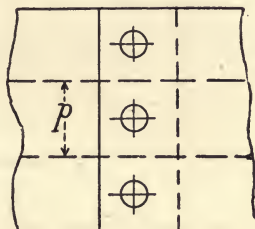


Fig. 253.

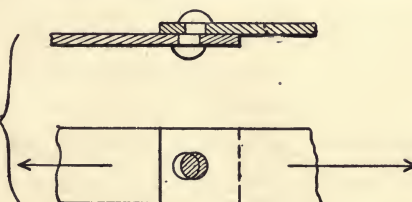


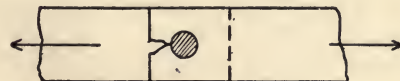
Fig. 254.



Fig. 255.



Fig. 256.



2. The plate may tear along the line of minimum section (Fig. 254). The area of either plate on this line is  $(p-d)t$ . The resistance to tension is

$$T = f_t (p-d)t \dots\dots\dots (2).$$

3. The plate and rivet may be crushed (Fig. 255), and this will render the joint loose. The area of plate or rivet supporting the pressure  $= dt$ ; this area is called the *bearing area*, and the pressure upon it the *bearing pressure*. The resistance to crushing is

$$T = f_c dt \dots\dots\dots (3).$$

4. The plate may break in front of the rivet (Fig. 256). The portion of plate in front of the rivet may be considered as a beam of length  $d$ , and depth  $= \frac{1}{2}(t-d)$ . Suppose the pull  $T$  to be replaced by two parts  $\frac{T}{2}$ , each acting half-way between the centre and edge of the rivet. This gives a bending moment of  $\frac{Td}{8}$ , and equating this to the moment of resistance

$$\frac{Td}{8} = f_t Z = \frac{f_t t \left(l - \frac{d}{2}\right)^2}{6};$$

or

$$T = f_t \frac{t(2l-d)^2}{3d} \dots\dots\dots (4).$$

The resistances to shearing, tearing, crushing, or breaking should be equal.

When the rivets are in double shear equation (1) becomes

$$T = f_s \frac{\pi d^2}{2}.$$

### 109. Resistance of multiple-riveted joints.

When there are more than two rows of rivets parallel to the joint in each plate it is called *multiple riveted*.

Let  $T$  = total longitudinal force, transmitted through the joint.

„  $n$  = number of rivets required in each plate joined, that is the total number through the joint if a lap joint, or the number on each side of the joint in a butt joint.

Then, assuming that  $T$  is *uniformly distributed among the  $n$  rivets*,  $n$  must be such that

$$T = \begin{cases} n f_s \frac{\pi d^2}{4} & \text{for rivets in single shear;} \\ n f_s \frac{\pi d^2}{2} & \text{for rivets in double shear;} \end{cases}$$

and

$$T = n f_c dt;$$

When there is more than one row of rivets in a lap joint or on one side of the joint in a butt joint, we

also if  $b$  = breadth of plate,  
 $m$  = number of rivets in one transverse row,

the tensile resistance of the net section of the plate is  $f_t(b - md)t$ , and it is necessary that the number  $m$ , and the dimensions  $b$ ,  $t$ , should be such that  $T = f_t(b - md)t$ .

# 110. Tensile, shearing, and crushing strength of plates and rivets in riveted joints.

The values given are the *breaking strengths*.

*Tensile strength of iron and steel plates* (unperforated),

Wrought iron, 18 to 24 tons per sq. inch.

Steel, 28 to 32 „ „

*Tensile strength of iron and steel in riveted joints.* The strength of the net section of plate appears to be increased by some causes and diminished by others.

One cause of loss is the injury done to the plate by punching, due chiefly to the pressure of the punch straining the metal round the hole, causing it to become harder and more brittle. The loss of strength depends chiefly on the thickness and quality of the plate, and is less in the case of wrought-iron plates than of steel plates. For soft wrought iron the loss is 4 to 8 per cent., while for harder plates it is 20 per cent., and for steel plates it is from 8 to 35 per cent.

The injury due to punching may be remedied by annealing or by rimering out the holes; in the latter case the hole is punched smaller than the intended diameter, and then rimered out about  $\frac{1}{8}$  inch all round.

Single-riveted lap joints, and butt joints with one cover only, are subject to a further loss of strength due to the tendency of such joints to straighten out, so that the resultant force may act in a line through the middle of the plates; consequently bending takes place, and the resistance of the joint is diminished.

Another cause of loss is the unequal distribution of stress, owing to its concentration at the edges of the pulling rivets; and this is emphasised when the bearing pressure is too great.

On the other hand, there appears to be a distinct gain in the tensile strength of the net section due to perforation, owing probably to the more uniform distribution of stress over the portions of the plate between the holes. by drilling

Experiment shows that the tensile strength of a drilled plate is 10 per cent. greater than the original undrilled plate, but this applies to boiler-work, where the pitch is small; the increase would be very much less in the case of girder work, where the pitch is usually large.

The following average values give the ratio of the tensile strength

of joint plate  $f_t$  to the tensile strength of the original unperforated plate :

					Iron Plates	Steel Plates
Single-riveted joints, punched	...	...			$\frac{3}{4}$	$\frac{9}{10}$
"      "      drilled	...	...			$\frac{7}{8}$	1
Double-riveted joints, punched	...	...			$\frac{7}{8}$	1
"      "      drilled	...	...			1	$\frac{10}{9}$

or the following average values of *the tensile strength*  $f_t$  of the different joints may be taken :

					Iron Plates	Steel Plates
Single-riveted, punched	...	...	...		16 Tons	25 Tons
"      drilled	...	...	...		18	28
Double-riveted, punched	...	...	...		18	28
"      drilled	...	...	...		20	30

*Shearing strength of iron and steel.*  $f_s$ . Shearing resistance, iron and steel plates, is approximately  $\frac{4}{5}$ ths of the tensile strength or  $f_s = \frac{4}{5} f_t$ . But for *riveted joints* the following values should be taken :

*Ratio of tensile and shearing resistance  $\frac{f_t}{f_s}$  in riveted joints (Unwin).*

Joints	Iron Plates, Iron Rivets		Steel Plates, Steel Rivets	
	Drilled	Punched	Drilled	Punched
Single-riveted ...	0.94	0.77	1.25	1.05
Double-riveted ...	1.02	0.85	1.34	1.17

*Crushing Pressure.*  $f_c$ . The crushing or bearing pressure should not exceed 40 to 43 tons per sq. inch. The relation between crushing strength and shearing strength is yet undetermined, but it has been found by experiment that when the crushing pressure amounted to 50 tons per sq. inch, the shearing strength of the rivets was reduced from 24 to 18 tons per sq. inch of rivet section. Thus 50 tons is taken as the limit, otherwise the rivets become too weak.

If we consider a joint where the rivets are in *single shear*, and call

$f_c$  and  $f_s$  the crushing and shearing stress, equating the crushing and shearing resistance we get

$$f_c dt = \frac{f_s \pi d^2}{4},$$

$$\frac{f_c}{f_s} = 0.785 \frac{d}{t} \dots\dots\dots(5),$$

or

$$\frac{d}{t} = 1.27 \frac{f_c}{f_s} \dots\dots\dots(6).$$

For rivets in double shear

$$\frac{d}{t} = 0.635 \frac{f_c}{f_s} \dots\dots\dots(7).$$

Thus we see that the crushing pressure increases as the ratio of diameter of rivet increases.

If we take a limiting ratio of  $\frac{f_c}{f_s} = 2$ , then from (6)  $d \leq 1.27 t (= 1.27 t)$  *call  $\frac{f_c}{f_s} = 2$*

$$d \geq 2.54 t \text{ for rivets in single shear } \dots\dots\dots(8), \quad t > \left(\frac{1.27}{1.27}\right)^2 = 0.8$$

and from (7)  $d \geq 1.27 t$  for rivets in double shear  $\dots\dots\dots(9).$  *cf. 7/8 below*

The diameters of rivets as calculated from Professor Unwin's rule  $d = 1.2 \sqrt{t}$  for different thicknesses of plates are given in tabular form.

Thickness of plate } in inches	$\frac{5}{16}$	$\frac{3}{8}$	$\frac{7}{16}$	$\frac{1}{2}$	$\frac{9}{16}$	$\frac{5}{8}$	$1\frac{1}{8}$	$\frac{3}{4}$	$1\frac{3}{8}$	$\frac{7}{8}$	$1\frac{5}{8}$	1
Diameter of rivets } in inches	$1\frac{1}{8}$	$\frac{3}{4}$	$1\frac{3}{8}$	$\frac{7}{8}$	$\frac{7}{8}$	$1\frac{5}{8}$	1	$1\frac{1}{8}$	$1\frac{3}{8}$	$1\frac{1}{8}$	$1\frac{3}{8}$	$1\frac{1}{4}$

From (8) and (9) we see that with rivets thus proportioned there is really no necessity to ever consider the crushing action in single-shear joints, and in double shear joints only when the plates are less than  $\frac{7}{8}$  inch.

#### RESISTANCE OF RIVETED JOINTS.

The strength of a riveted joint is greatest when it offers equal resistance to each of the four modes of failure described. The relative values of  $p$ ,  $d$ ,  $l$ ,  $t$  obtained from equations (1), (2), (3), (4) modified, if necessary, by practical considerations may be considered as good proportions for the joint.

#### 111. CASE 1. SINGLE-RIVETED LAP JOINT. SINGLE-RIVETED BUTT JOINT WITH ONE COVER.

See Figs. 245 and 246.

In both these cases the rivets are in single shear.

*Diameter of rivet.* As explained in the last article, by equating the shearing resistance to the bearing resistance we get too large a value for the diameter. This is fixed in terms of the thickness of

the plate  $t$  by the formula  $d = 1.2\sqrt{t}$ . The effect of taking this ratio of  $\frac{d}{t}$  in preference to the greater theoretical value is to diminish the shearing area as compared with the bearing area, thus increasing the bearing resistance of the rivet as compared with its shearing resistance.

*Pitch.* Equating the tearing resistance to the shearing resistance,

$$(p-d)tf_t = \frac{\pi d^2}{4} f_s = 0.785d^2 f_s;$$

$$p = 0.785 \frac{d^2}{t} \cdot \frac{f_s}{f_t} + d;$$

so the pitch  $p$  can be found by using the value of  $\frac{f_s}{f_t}$  given in table for these joints.

*Overlap.* Equating the tearing resistance of the plate in line of rivets to the tearing resistance in front of the rivet, equations (2), (4),

$$t(p-d) = \frac{(2l-d)^2}{3d} \cdot t;$$

$$(2l-d)^2 = 3d(p-d);$$

$$\therefore 2l = d + \sqrt{3d(p-d)}.$$

This must be at least three times the diameter of rivet.

CASE 2. DOUBLE-RIVETED LAP JOINT. DOUBLE-RIVETED BUTT JOINT WITH SINGLE COVER.

See Figs. 248 and 249.

Since there are in this case two rivets to each strip of a width equal to the pitch, equations (1) and (3) become

$$T = \frac{\pi d^2}{2} f_s;$$

$$T = 2f_c dt.$$

*Diameter.* The values of the ratio  $\frac{d}{t}$  got from these two equations would be the same as in Case 1, and  $d$  is fixed as before from the formula  $d = 1.2\sqrt{t}$ .

*Pitch.* Equating the tearing and shearing resistance here,

$$(p-d)tf_t = \frac{\pi d^2}{2} f_s;$$

$$p = 1.57 \frac{d^2}{t} \frac{f_s}{f_t} + d.$$

From this the pitch can be found by taking the value of ratio  $\frac{f_s}{f_t}$  from table for double-riveted joints.

*Overlap.* In chain-riveted joints it is best to allow  $1\frac{1}{2}$  times the

diameter of the rivet between the edges of the holes in the two rows, especially when the holes are punched. The distance between the pitch lines (centre lines of each row of rivets) is then  $2\frac{1}{2}$  times the diameter, and the overlap becomes  $5\frac{1}{2}$  diameters.

In zig-zag riveted joints (Figs. 249 and 251) it is necessary that the distance between the pitch lines should be such that the resistance of the plate to fracture along the zig-zag line is at least equal to the resistance to fracture along either pitch line. In practice the distance between the pitch lines is usually taken  $\frac{3}{4}$  of the pitch,

so the lap = 3 diameters +  $\frac{3}{4}$  pitch.

### CASE 3. SINGLE-RIVETED BUTT JOINT WITH TWO COVERS.

See Fig. 246.

For this form of joint the rivets are in double shear, and equation (1) for shearing resistance becomes  $\frac{\pi d^2}{2} f_s$ , the other equations remaining the same.

*Diameter.* Equating the shearing resistance to bearing resistance,

$$\frac{d}{t} = \frac{2 f_c}{\pi f_s};$$

if this gives a smaller value than the empirical rule for  $\frac{d}{t}$  the latter must be taken. The effect of taking the larger value is to increase the shearing resistance of the rivet as compared with its bearing resistance.

To determine the *pitch* and *lap* for this case, the resistances of the plate to direct stress upon its net section, and to tearing out, have therefore to be equated to the least resistance of the rivet, that is its bearing resistance.

### CASE 4. DOUBLE-RIVETED BUTT JOINT WITH TWO COVERS.

See Figs. 250 and 251.

With this joint there are two rows of rivets in double shear, and equations (1) and (2) become

$$T = f_s \pi d^2,$$

$$T = 2 f_c d t.$$

*Diameter. Pitch.* The ratio of  $\frac{d}{t}$  got from these equations would be the same as in Case 3, and the same remarks apply.

*Lap.* Same as in Case 2.

In chain-riveted joints distance between pitch lines is  $2\frac{1}{2}$  diameters. Lap =  $5\frac{1}{2}$  diameters.

For zig-zag riveted joints, lap = 3 diameters +  $\frac{3}{4}$  of the pitch.

It must be remembered that all the values given for the *lap* are

minimum values for good workmanship. With ordinary work, especially when the holes are punched and the edges of the plates not planed, an addition of 10 to 15 per cent. should be made.

### 112. Thickness of cover plates.

The thickness of the cover plates must be such that the strength of their net section is *at least* equal to that of the net section of the plates to be joined.

The usual proportions are:

With one cover plate, thickness =  $1\frac{1}{8}$  of the plate thickness.

With two cover plates, thickness of each =  $\frac{5}{8}$  of the plate thickness.

### 113. Efficiency of riveted joints.

The *efficiency* of a joint is the ratio of the strength of the joint to the strength of an equal width of the solid plate.

Taking a strip of the joint of width equal to the pitch  $p$ ,

$$\text{Efficiency} \propto \frac{p-d}{p} = k \left( \frac{p-d}{p} \right),$$

where  $k = \frac{\text{tensile strength of the net section of joint}}{\text{tensile strength of solid plate}}.$

The strength of joint =  $k \left( \frac{p-d}{p} \right) \times$  the strength of the solid plate.

The following approximate values of  $k$  can be taken:

	Iron		Steel	
	Punched	Drilled	Punched	Drilled
Single-riveted joints ...	$\frac{3}{4}$	$\frac{7}{8}$	$\frac{7}{8}$	$\frac{9}{10}$
Double-riveted joints ...	$\frac{7}{8}$	$\frac{9}{10}$	$\frac{9}{10}$	1

### 114. Group-riveted joints.

Joints are sometimes made by a group of rivets, so arranged that as little as possible of the original resistance of the unperforated plate is lost at the joint. The arrangement of the rivets is called *group riveting*.

In order to get the stress uniformly distributed over the plate the centre of gravity of the group of rivets must lie on the axis of the piece, the axis being the line joining the centres of gravity of the cross sections. When two plates not in line are to be riveted, as in the bracing and the flange of a girder, the centre of gravity of the group ought to lie on the intersection of the axes of the two plates.

### 115. Group-riveted joint of greatest economy.

In an ordinary group-riveted joint (Fig. 257) the net section of the plate is its gross section diminished by all the rivet holes in the transverse row nearest to the end of the cover plate. By adopting

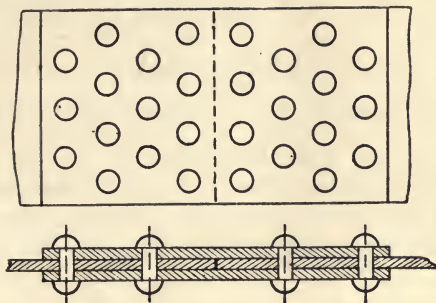


Fig. 257.

the form shown in Fig. 258 the loss of section may be reduced to that due to one rivet hole only.

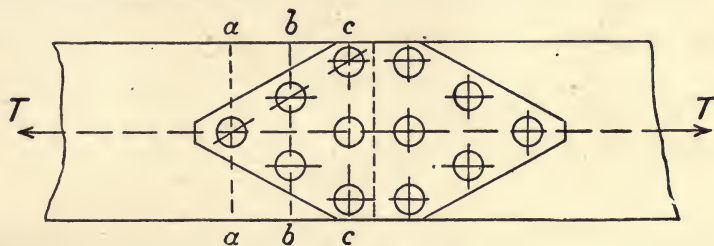


Fig. 258.

Consider, for example, a joint for which calculation gives  $n = 6$  rivets required on each side of joint.

A single rivet is placed in the line  $aa$  on the axis of the plate, diminishing the section by one rivet hole, and on the net section we have the whole stress  $T$ . Now, assuming that the stress  $T$  is equally distributed between the 6 rivets in the groups the leading rivet transmits  $\frac{T}{6}$  to the cover plates, so that the stress on the net section at  $bb$  is  $\frac{5}{6}T$ . A second rivet may therefore be placed at  $bb$  without diminishing the resistance of the joint. At section  $cc$  the stress is  $\frac{1}{2}T$ , so that one more rivet may be placed at that section.

At  $cc$  the stress in the cover plates will be equal to  $T$ .

The distances  $ab$ ,  $bc$  are usually  $\frac{3}{4}$  of the transverse pitch.

The strength of the joint is approximately equal at all the sections and may be taken as

$$(b - md) f_t \cdot t.$$

The thickness of the cover plates must be such that the resistance of

their net section at the transverse row of rivets  $cc$  nearest to the joint is at least equal to the stress  $T$ .

Let  $t_1$  = required thickness of each cover plate ;

„  $m_1$  = number of rivets in row  $cc$  ; then

$$2t_1(b - m_1d) \cdot f_t \equiv T,$$

or

$$t_1 \equiv \frac{T}{2(b - m_1d)f_t}.$$

The width of the cover plates is tapered uniformly as in Fig. 258.

### Examples.

1. A single-riveted lap joint for a pair of steel plates  $\frac{1}{2}$ " thick has to transmit a tensile stress of 30 tons. Determine the diameter, pitch, number of rivets, and width of plate required for the joint. Rivet holes drilled.

Take  $f_t = 28$  tons per sq. inch }  
 „  $f_s = 22$  „ „ „ } breaking stress.  
 „  $f_c = 44$  „ „ „ }

*Diameter.* Equate bearing resistance to shearing resistance,

$$f_c dt = f_s \frac{\pi d^2}{4},$$

$$d = 2.4t.$$

This diameter is too great.

Take  $d$  according to rule :

$$d = 1.2 \sqrt{t} \\ = \frac{7}{8}'' ,$$

as the rivet holes are made about 4 per cent. larger than the diameter of the rivet, and in riveting up the rivet is compressed to fill the hole. The rivets will, in all examples, be taken 4 per cent. larger than their nominal diameter.

*Pitch.* Diameter  $\frac{7}{8}$ " + 4 per cent. = 0.91".

Equating the tearing and shear-resistances,

$$(p - d) t f_t = 785 d^2 f_s,$$

$$p = 785 \frac{d^2 f_s}{t f_t} + d \\ = 785 \times 1.33 + 0.91 \\ = 1.96'', \text{ say } 2''.$$

*Number of rivets.* The working

shear stress may be taken at 5 tons per sq. inch, giving a factor of safety of about  $4\frac{1}{2}$ .

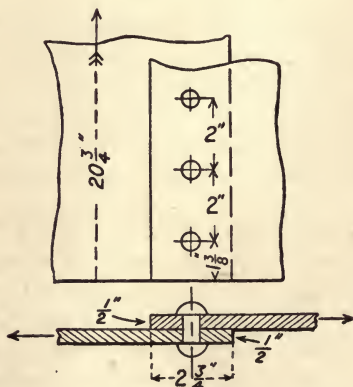


Fig. 259.

The shearing resistance of each rivet is

$$f_s \frac{\pi d^2}{4} = 5 \times 0.6 = 3 \text{ tons,}$$

which is less than the bearing resistance.

Therefore number of rivets required is

$$N = \frac{30 \text{ tons}}{3 \text{ tons}} = 10.$$

*Width of plate.* Let  $b$  = width required.

The tensile *working* stress may be taken as 6 tons per sq. inch. Then along a line of rivets we must have

$$T \geq f_t (b - 10d) t,$$

or,

$$30 \geq 6 (b - 9.1) \frac{1}{2},$$

$$\therefore b \geq 19.1''.$$

This value is not sufficient for  $b$ , as it must be in this case at least equal to 9 times pitch + 3 diameters of rivets.

Take  $b = 9 \times 2 + 3 \times .91 = 20.73 = 20\frac{3}{4}''$ , which allows a distance of  $\frac{3}{2}d$  from the centre of the nearest rivet to each lateral edge of the plate.

*Efficiency.* The efficiency of the joint

$$\begin{aligned} &= k \left( \frac{p-d}{p} \right) \\ &= \frac{9}{10} \left( \frac{1.96 - .91}{1.96} \right) = 0.5. \end{aligned}$$

See Fig. 259 for plan and section of joint.

2. Two steel plates  $\frac{3}{4}''$  thick are to be connected by a double-riveted lap joint. The joint has to resist a tensile force of 60 tons. Find the diameter, pitch, and number of rivets required for the joint, the holes being drilled.

$$\left. \begin{array}{ll} \text{Take } f_t = 28 \text{ tons per sq. inch} \\ \text{,, } f_s = 24 \text{ ,, } \\ \text{,, } f_c = 48 \text{ ,, } \end{array} \right\} \text{breaking stress.}$$

*Diameter.* Take  $d = 1.2 \sqrt{t} = 1\frac{1}{8}''$ ;

or  $d$  for calculation  $= 1\frac{1}{8} + 4\frac{9}{16} = 1.1''$ .

*Pitch.* Equate shearing and tearing resistances,

$$t(p-d)f_t = \frac{\pi d^2}{2} f_s;$$

$$p = 1.57 \frac{d^2 f_s}{t f_t} + d$$

$$= 1.57 \frac{1.21 \times 4}{3} \times 2 + 1.1 = 3.2'' = 3\frac{1}{4} \text{ ins.}$$

$$1.57 \frac{1.21 \times 4}{3} = 3.2 - 1.1 \quad x = .83 ?$$

$\frac{f_s}{f_t} = .746$   
 $\frac{1}{1.34} \sim \frac{f_s}{f_t}$   
*becomes*

*Number of rivets.* Taking the *working* shear stress at 5 tons per sq. inch,

$$\begin{aligned}\text{Shearing resistance of one rivet} &= f_s \frac{\pi d^2}{4} = 5 \times 0.95 \\ &= 4.75 \text{ tons.}\end{aligned}$$

$$\begin{aligned}\text{Therefore number of rivets} &= \frac{60}{4.75} = 12.6 \\ &= 13, \text{ say.}\end{aligned}$$

*Width of plate.* If we arrange the rivets in two rows of 6 rivets and 7 rivets respectively,

Let  $b$  = width of plate along the line of the 7 rivets, then

$$t(b - 7d)f_t \leq 60 \text{ tons,}$$

where  $f_t$  = *working* tensile stress per sq. inch,  
=  $5\frac{1}{2}$  tons per sq. inch (say),

$$\begin{aligned}\text{then} \quad \frac{3}{4}(b - 7 \cdot 7) 5\frac{1}{2} &\leq 60, \\ b &\leq 22.2''.\end{aligned}$$

The width has to be slightly greater in order to get  $1\frac{1}{2}d$  from the centre of nearest rivet to edge of plate, consequently  $b$  is made

$$= 6p + 3d = 22.5''.$$

$$\text{Efficiency} = k \left( \frac{p-d}{p} \right) = 1 \left( \frac{3.2 - 1.1}{3.2} \right) = 0.65.$$

*Lap.* Distance from centre of each row of rivets to end of plate =  $1\frac{1}{2}d = 1.65$ , say  $1\frac{3}{4}''$ ; and between pitch lines =  $\frac{3}{4}p = 2\frac{1}{2}''$ . The lap is therefore  $2\frac{1}{2} + 2 \times 1\frac{3}{4} = 6''$ .

Fig. 260 shows the joint in plan and section.

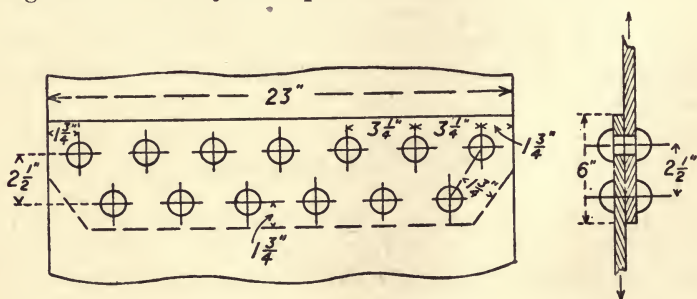


Fig. 260.

3. *Required, the dimensions of a zig-zag double-riveted butt joint with two covers for two wrought iron boiler-plates  $\frac{3}{8}''$  thick, the rivet holes being punched (Fig. 261).*

Take  $f_t = 22$  tons per sq. inch.

„  $f_s = 18$  „ „  
„  $f_c = 36$  „ „

There are two rows of rivets in double shear.

$$T = f_s \pi d^2 \dots \dots \dots (1).$$

$$T = 2f_c dt \dots \dots \dots (2).$$

$$T = (p - d) t f_t \dots \dots \dots (3).$$

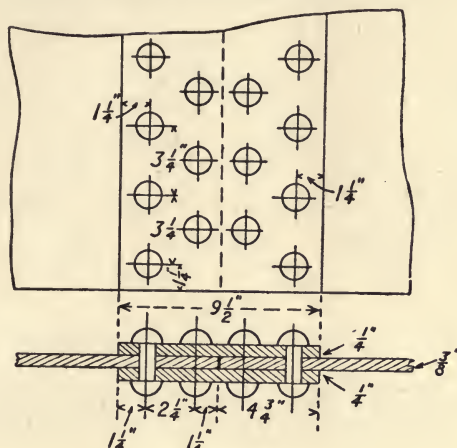


Fig. 261.

*Diameter.* Equating (1) and (2),

$$f_s \pi d^2 = 2f_c dt,$$

$$d = \frac{2}{\pi} \frac{f_c}{f_s} t = \frac{14}{22} \times 2 \times \frac{3}{8} = 0.47.$$

This value of  $d$  is *too small*.

Take  $d = 1.2 \sqrt{t} = \frac{3}{4}$ "; that is, diameter for calculation =  $\frac{3}{4} + 4$  per cent. = 0.78".

The effect of thus *increasing* the diameter is to diminish the intensity of shearing stress upon the section of a rivet, and so increasing the shearing resistance as compared with its bearing resistance.

*Pitch.* To get the *pitch* equate the tearing resistance to the least resistance of a rivet; that is, its bearing resistance instead of, as before, to its shearing resistance.

$$(p - d) t f_t = 2f_c dt,$$

$$p = d \left\{ 1 + 2 \frac{f_c}{f_t} \right\} = 0.78 \left\{ 1 + \frac{36}{11} \right\} = 3 \frac{1}{4}."$$

*Lap*, or distance between the edges of the plate and cover plate,

$$\begin{aligned} l &= 1 \frac{1}{2} d + \frac{3}{4} p + 1 \frac{1}{2} d \\ &= 1 \frac{1}{4}" + 2 \frac{1}{4}" + 1 \frac{1}{4}" \text{ (app.)} = 4 \frac{3}{4}". \end{aligned}$$

*Cover plates.*

$$\text{Width} = 2l = 9 \frac{1}{2}."$$

$$\text{Thickness} = \frac{5}{8} t = \frac{1}{4}."$$

$$\text{Efficiency} = k \left( \frac{p-d}{p} \right) = 0.9 \left( \frac{3\frac{1}{4} - 0.78}{3\frac{1}{4}} \right) = 0.69, \text{ or } 69 \text{ per cent.}$$

4. A tie-bar  $\frac{1}{2}$ " thick has to transmit a tensile stress of 36 tons. Design a butt joint with two cover plates, such that not more than one rivet hole is lost from the gross section of the plate (see Fig. 262).

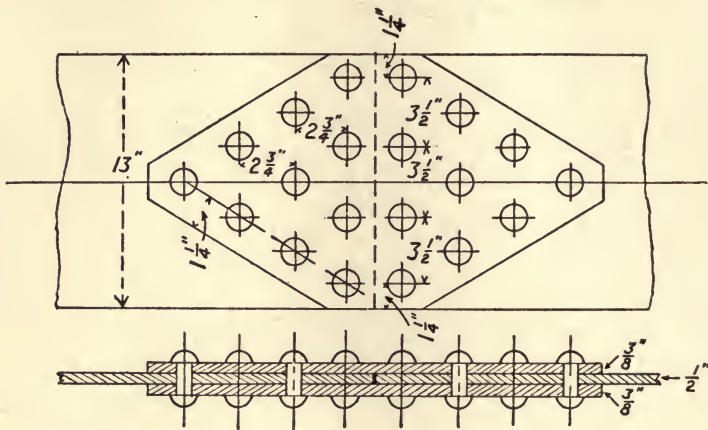


Fig. 262.

Working intensities of stress are

$$f_t = 6 \text{ tons per sq. inch.}$$

$$f_s = 4\frac{1}{2} \quad \text{,,} \quad \text{,,}$$

$$f_c = 9 \quad \text{,,} \quad \text{,,}$$

Diameter of rivets  $\frac{3}{4}$ ".

Number of rivets. Let  $N$  = number of rivets required.

$$\text{Then} \quad N f_s \frac{\pi d^2}{2} = 36,$$

$$N \times 4.5 \times 0.88 = 36,$$

$$N = 9. \quad \text{Say } 10.$$

Width of plate. Let  $b$  = width of plate.

$$\text{Then} \quad (b-d) \cdot t \cdot f_t = 36 \text{ tons,}$$

$$(b - \frac{3}{4}) \cdot \frac{1}{2} \cdot 6 = 36,$$

$$b = 12\frac{3}{4} \text{".} \quad \text{Say } 13 \text{".}$$

The rivets are arranged as in Fig. 262, which represents the joint in plan and section.

Thickness of cover plates. Let  $t$  = thickness of each cover plate.

Then, taking the section along the line of four rivets,

$$2t(b-4d)f_t = 36,$$

$$2t(13 - 4 \times \frac{3}{4})6 = 36,$$

$$t = \frac{12}{33} = 0.36 \text{"} = \frac{3}{8} \text{", say.}$$

5. A tie-bar 6" by  $\frac{1}{2}$ " transmits a stress of 12 tons. Find the number of  $\frac{3}{4}$ " rivets required to connect it to the side plate of a girder boom made up of  $\frac{1}{2}$ " plates and angle-irons (see Fig. 263).

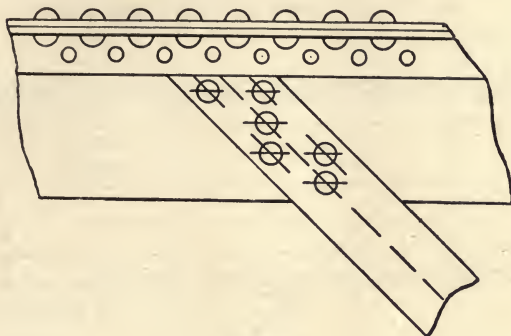


Fig. 263.

Net section of bar, deducting one rivet hole

$$= (6 - \frac{3}{4}) \frac{1}{2} = 2.62 \text{ sq. inches.}$$

$$\text{Intensity of stress} = \frac{12}{2.62} = 4.6 \text{ tons per sq. inch.}$$

Let  $N$  = number of rivets required,

$f_s = 5$  tons per sq. inch, then

$$\begin{aligned} 12 &= N \frac{\pi d^2}{4} \times f_s \\ &= N \times 0.44 \times 5, \\ \therefore N &= 5.4. \text{ Say } 6. \end{aligned}$$

The six rivets must be *symmetrically* grouped round the centre line or axis of the bar, one rivet being in the first row, as we have assumed that the bar is only weakened by one rivet hole.

Very often it is necessary to connect the ties to the boom by means of subsidiary plates, called gusset plates or connecting plates, of which Fig. 264 is an example.

6. A tie, 14" by  $\frac{3}{4}$ ", has to be joined to the side plate of a girder boom by a connecting plate  $\frac{3}{4}$ " thick, using a butt joint with two cover plates. Design a suitable joint for the connection, arranging that the tie is weakened only by one rivet section (Fig. 264).

Tensile stress in tie = 62 tons.

Diameter of rivets,  $\frac{3}{4}$ ".

Working shear stress = 5 tons per sq. inch. ✓

Let  $N$  = number of rivets required. ✓

The total shearing area of the rivets on each side of the joint line multiplied by the safe working shear stress for the rivets should equal the total stress transmitted through the plates.

Then, as the rivets are in double shear, we have

$$T = N \times 2 \times \frac{\pi d^2}{4} \times f_s,$$

i.e.,  $62 = N \times 2 \times 0.44 \times 5 = 14.1$ , say 15,  
arranged as in Fig. 264.

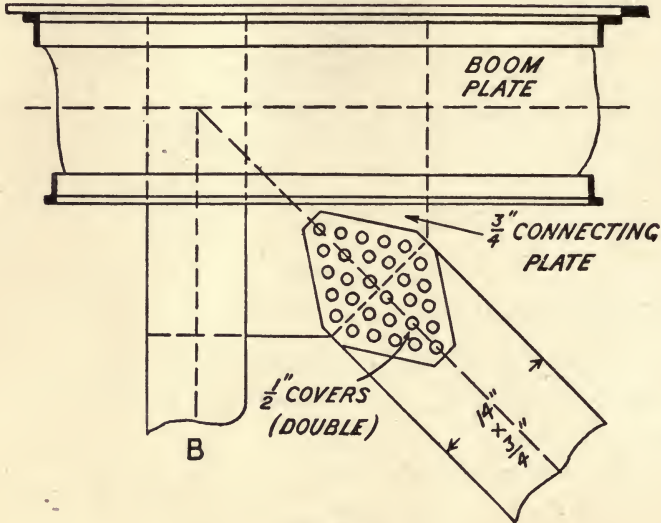


Fig. 264.

Or, we can find the number of rivets by comparing the rivet area and net cross sectional area of plate *previously designed*, thus :

$$\frac{\text{The rivet area}}{\left( \begin{array}{l} \text{Sectional area} \\ \text{of plate, less} \\ \text{end rivet hole} \end{array} \right)} = \frac{\text{tensile or compressive strength of plate per sq. inch}}{\text{shearing strength of rivet steel per sq. inch}}.$$

The following values for steel may be taken :

Tensile strength of steel 27 to 30 tons per sq. inch.

Compressive strength of steel 21 tons per sq. inch.

Shearing strength of rivets 20 tons per sq. inch.

In above example, as the rivets are in double shear,

$$\begin{aligned} N \times 2 \times \frac{\pi d^2}{4} &= \frac{27}{20}, \\ 14 \times \frac{3}{4} - \frac{3}{4} \times \frac{3}{4} &= \frac{27}{20}, \\ N &= \frac{27}{20} \cdot \frac{(14 \times \frac{3}{4} - \frac{3}{4} \times \frac{3}{4})}{2 \times 0.44} = 15. \end{aligned}$$

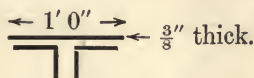
*Cover plates.* Each cover plate is in this case weakened by 5 rivet holes.

Let  $t$  = thickness of one cover plate, then

$$2t(14 - 5 \times \frac{3}{4}) = (14 - \frac{3}{4}) \frac{3}{4},$$

$$t = \frac{10}{20 \cdot 5} = \frac{1}{2} \text{ inch, say.}$$

7. To find the number of  $\frac{3}{4}$ " rivets required to connect the vertical compression post (B) (Fig. 264) to the side plate of boom.



Let the area of cross section of post as designed consist of

$$1 \text{ plate } 12'' \times \frac{3}{8}'' = 4 \cdot 5 \text{ sq. inches.}$$

$$2 \text{ angles } 4'' \times 3\frac{1}{2}'' \times \frac{3}{8}'' = 5 \cdot 38 \quad ,,$$

$$\underline{9 \cdot 88} \quad ,,$$

$$\text{Less rivet holes} \quad \underline{1 \cdot 70} \quad ,,$$

$$\text{Net area} \quad \underline{\underline{8 \cdot 18}} \quad ,,$$

The rivets being in single shear, the number of rivets required

$$= \frac{\text{net area of strut}}{\text{area of one rivet}} \times \frac{21}{20} = \frac{8 \cdot 18}{0 \cdot 44} \times \frac{21}{20} \\ = 20 \text{ (nearly).}$$

*Joint for two or more plates.* When several plates have to be riveted together, their joints are arranged in consecutive steps as in



Fig. 265.

Fig. 265, so that one pair of cover plates is sufficient for the whole series of joints. The length of lap is generally twice the longitudinal pitch of the riveting. The number of rivets between any two consecutive joints must be proportioned to the stress. The stress in the cover plates is that given to them by the rivets.

## 116. Thin shells or boilers.

CIRCULAR OR HOOP STRESS IN A THIN CYLINDRICAL SHELL.

LONGITUDINAL JOINTS OF A BOILER.

Let Fig. 266 represent the section of a cylindrical boiler ; and let

$r$  = internal radius in inches,

$t$  = thickness of shell in inches, always very small compared with  $r$ .

$p$  = intensity of internal pressure in lbs. per sq. inch, acting normally to the surface.

Consider a portion of the cylinder  $l$  inches long.  
Then the resultant of the internal pressure is

$$P = 2rlp,$$

which must be in equilibrium with the total tensile stress on the section of the plates at  $A$  and  $B$ .

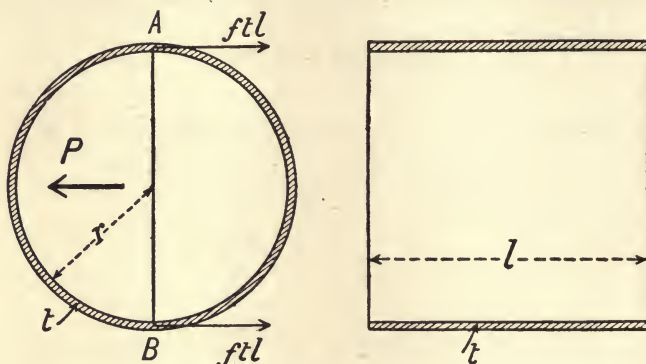


Fig. 266.

If  $f$  lbs. per sq. inch is the intensity of tensile stress, assumed uniform, as the thickness of plates is small, then  $2(t \times l)$  sq. inches is the area of metal cut through at the section  $AB$ , and for equilibrium we must have

$$2tlf = 2rlp.$$

Hence

$$f = \frac{pr}{t} \dots\dots\dots(1),$$

or

$$p = \frac{ft}{r} \dots\dots\dots(2).$$

Equation (2) gives the working pressure if  $f$  is the working stress, and is the formula always used for the strength of a cylindrical pipe or boiler. It should be noted that a boiler being composed of plates riveted together, the strength  $f$  must be taken as that of the metal in the joint, and not that of the solid metal.

As stated in Art. 113,

The strength of joint = efficiency of joint  $\times$  strength of the solid metal.  
Consequently, if  $f$  is taken to denote the strength of the solid metal, equation (1) becomes

$$f \times \text{efficiency of joint} = \frac{pr}{t},$$

and

$$t = \frac{pr}{f \times \text{efficiency of joint}}.$$

This gives the necessary thickness of the plates.

$f$  may be taken as 10,000 lbs. per sq. inch for wrought iron, and 12,000 lbs. per sq. inch for mild steel.

### LONGITUDINAL STRESS ON A TRANSVERSE SECTION.

Let Fig. 267 represent a longitudinal diametral section of the cylinder;  $CD$  a transverse section.

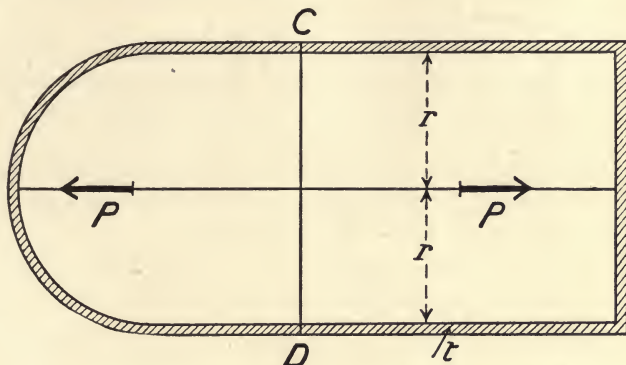


Fig. 267.

Consider the equilibrium of either portion of cylinder.

The resultant internal pressure acting along the axis of cylinder upon the end, which may be either curved or flat, is

$$P = p \times \text{area of section of the shell} \\ = p \times \pi r^2,$$

which is balanced by the longitudinal stress,  $f'$  say, exerted on the cross section of the shell at  $CD$ , namely, the ring whose area is  $2\pi r \cdot t$ .

Hence

$$p \cdot \pi r^2 = f' 2\pi r t,$$

$$f' = \frac{pr}{2t} \dots\dots\dots(3),$$

or

$$p = \frac{2f't}{r} \dots\dots\dots(4).$$

If the ends are connected by stays this relation does not hold, as the stays relieve the shell plates from any longitudinal tension.

### EXERCISES.

1. Determine the thickness of the plates suitable for a boiler 6 feet diameter, working under a pressure of 100 lbs. per square inch. Take the efficiency of the joints to be 70 per cent., and the safe stress as 10,000 lbs. per square inch. *Ans.*  $\frac{1}{2}$  inch.

2. Two mild-steel tie-rods 8 inches by  $\frac{7}{8}$  inch are to be connected by a butt joint with double straps. Design the joint and calculate the efficiency.

3. The steel plates of a boiler are  $\frac{1}{2}$  inch thick, connected by longitudinal double-riveted butt joints, with two covers or straps. Determine the diameter of the rivets, pitch, and efficiency.

4. The plates of a locomotive boiler, 4 feet diameter, are  $\frac{1}{2}$  inch thick. If the rivets are  $\frac{3}{4}$  inch diameter and  $1\frac{3}{4}$  inch pitch, determine the working pressure of the steam, taking the safe stress as 12,000 lbs. per square inch.

5. A boiler 6 feet diameter, of mild-steel plates, is subjected to a pressure of 120 lbs. per square inch. Design a double-riveted butt joint with single strap suitable for the longitudinal joint,

6. The steel plates of a cylindrical boiler 6 feet diameter are  $\frac{3}{4}$  inch thick. The plates are connected by a butt joint with two cover plates and are treble riveted. The holes are drilled. Determine the pitch and diameter of the rivets and the working pressure.

7. A pipe is 3 feet diameter and  $\frac{1}{2}$  inch thick. The working stress is 5 tons per square inch, but the strength of plate is reduced by 25 per cent. on account of riveted joint. Determine the working pressure.

*Ans.* 233.3 lbs. per sq. inch.

8. A tension plate of designed section  $1' 8'' \times \frac{1}{2}''$  has to be joined to a gusset plate  $\frac{5}{8}''$  thick by  $\frac{3}{4}''$  rivets. Find the number of rivets necessary, and design the joint so that the plate is weakened by only one rivet-hole. Double covers.

### 117. Strength of thick cylinders.

In considering the thin cylinder we neglected the variation of stress in the plates, and obtained the equation

$$p = \frac{ft}{r}.$$

If the thickness of the plates is considerable compared with the radius, the tensile stress can no longer be regarded as having the same intensity from inside to outside.

Let the internal and external radii of the cylinder (Fig. 268) be  $R_0$  and  $R_1$  respectively. Consider a ring of metal 1" in width parallel to the axis of the cylinder, of internal radius  $r$  and thickness  $dr$ . Let  $p$  be the intensity of the radial pressure on the inner surface of the ring,  $p + dp$  the intensity of the radial pressure on the outer surface, and call  $p'$  the compressive stress in the material at right angles to the radius (the hoop stress).  $p'$  has a negative value—that is, the stress is tensile—when the pressure inside the cylinder exceeds the

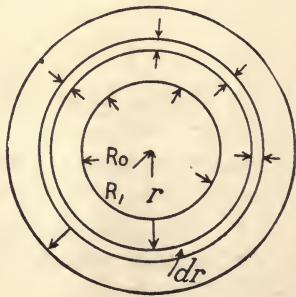


Fig. 268.

pressure outside. For equilibrium we equate the resultant radial force, tending to produce fracture, to the resultant of the forces due to hoop stress, which prevent fracture; that is

$$(p + dp) \times 2(r + dr) - p \times 2r = 2p'dr,$$

$$(p + dp)(r + dr) - pr = p'dr,$$

or 
$$r \frac{dp}{dr} + p = p' \dots \dots \dots (1).$$

We have obtained only a relation between the *stresses*. We require another equation which will express the method in which the cylinder yields.

Assume that plane cross sections remain plane when strained, that is, that the *longitudinal strain is uniform*. If we assume that the cylinder is subjected to a uniform longitudinal stress  $p_3$ , then, if  $\lambda_3$  is the longitudinal strain,

$$E\lambda_3 = p_3 - \frac{p + p'}{m},$$

and since  $p_3$  and  $\lambda_3$  are both constant,  $p + p'$  must be constant.

Take 
$$p + p' = 2c \dots \dots \dots (2).$$

Substituting in (2) the value of  $p'$  from equation (1),

$$r \frac{dp}{dr} + p = 2c - p,$$

or 
$$r \frac{dp}{dr} + 2p = 2c.$$

To solve this write it as

$$r^2 dp + 2pr dr = 2c r dr,$$

$$\frac{d}{dr} (pr^2) = 2cr.$$

Integrating, 
$$pr^2 = cr^2 + c_1,$$

or 
$$p = c + \frac{c_1}{r^2},$$

and 
$$p' = c - \frac{c_1}{r^2}.$$

The two constants depend on the pressure on the interior or exterior, and on the internal and external diameters.

### 118. Thick cylinder subjected to internal pressure only.

Let  $p_0$  be the internal pressure, and let the internal and external radii be  $R_0$  and  $R_1$  respectively, when

$$r = R_0; \quad p = p_0,$$

therefore

$$p_0 = c + \frac{c_1}{R_0^2},$$

when

$$r = R_1; \quad p = 0,$$

that is,

$$0 = c + \frac{c_1}{R_1^2}.$$

Subtracting,

$$p_0 = c_1 \left( \frac{1}{R_0^2} - \frac{1}{R_1^2} \right),$$

or

$$c_1 = \frac{p_0 (R_1^2 R_0^2)}{R_1^2 - R_0^2},$$

and

$$c = -\frac{p_0 R_0^2}{R_1^2 - R_0^2}.$$

Hence

$$p = -\frac{p_0 R_0^2}{R_1^2 - R_0^2} + \frac{p_0}{r^2} \frac{R_1^2 R_0^2}{R_1^2 - R_0^2},$$

$$p' = -\frac{p_0 R_0^2}{R_1^2 - R_0^2} - \frac{p_0}{r^2} \frac{R_1^2 R_0^2}{R_1^2 - R_0^2}.$$

These equations may be written

$$p = \frac{p_0 R_0^2}{R_1^2 - R_0^2} \left[ \frac{R_1^2}{r^2} - 1 \right] \dots\dots\dots (3),$$

and

$$p' = -\frac{p_0 R_0^2}{R_1^2 - R_0^2} \left[ \frac{R_1^2}{r^2} + 1 \right] \dots\dots\dots (4).$$

*The negative sign in the latter equation shows that  $p'$ , the hoop stress, is tensile.*

The hoop tensile stress has its maximum value at the inner surface, where  $r = R_0$ .

Thus

$$\text{Max. } p' = -p_0 \left[ \frac{R_1^2 + R_0^2}{R_1^2 - R_0^2} \right].$$

If  $f$  is safe tensile stress which the material of cylinder will bear, then the safe internal pressure is

$$\frac{f (R_1^2 - R_0^2)}{R_1^2 + R_0^2}.$$

### 119. Thick cylinder subjected to external pressure only.

If  $p_1$  is the external radial pressure, there being no internal pressure, the constants are

$$c = \frac{p_1 R_1^2}{R_1^2 - R_0^2},$$

$$c_1 = -\frac{p_1 R_1^2 R_0^2}{R_1^2 - R_0^2},$$

and the hoop stress at any radius  $r$  is

$$p' = \frac{p_1 R_1^2}{R_1^2 - R_0^2} \left[ \frac{R_0^2}{r^2} + 1 \right] \dots\dots\dots (5).$$

This pressure is a maximum at the inner surface, where  $r = R_0$ .

$$\text{Max. } p' = \frac{2p_1 R_1^2}{R_1^2 - R_0^2}.$$

## 120. Strength of thick cylinders when the material is initially strained.

If, before the cylinder is subjected to internal or external pressure, there already exist initial stresses in the material, then the stresses in (4) or (5) must be added algebraically to those already existing.

Thus, large guns are built up of two or more tubes. The outer tube, being heated and shrunk on to the inner one, produces a compressive stress in the inner tube and a tensile stress in itself.

Now, when an internal pressure is applied to this compound cylinder, the hoop tension it produces is added algebraically to the already existing hoop stresses, with the result that the stress on the outer portion is increased, and that on the inner portion is diminished, since its hoop tension is reduced by the initial stress. The distribution of stress is equalised.

### Examples.

1. *The external and internal diameters of a cylindric hydraulic press are 16" and 8" respectively. If the internal pressure is 3 tons per sq. inch, find the stresses at the inside and outside.*

$$R_1 = 8; \quad R_0 = 4.$$

Therefore, from equations (3) and (4),

$$p = \frac{3 \times 16}{48} \left[ \frac{64}{r^2} - 1 \right];$$

$$p' = -\frac{3 \times 16}{48} \left[ \frac{64}{r^2} + 1 \right].$$

At the *inner* surface  $r = 4$ ;

$$\therefore p = 3 \text{ tons per sq. inch};$$

$$p' = -5 \text{ tons per sq. inch.}$$

At the *outer* surface  $r = 8$ ;

$$\therefore p = 0;$$

$$p' = -2 \text{ tons per sq. inch.}$$

2. *A tube of 12" internal diameter and 30" external diameter is subjected to an internal pressure of 15 tons per sq. inch. Find the stress at points 1" apart radially between the inner and outer surfaces*

Here

$$R_1 = 15, \quad R_0 = 6;$$

$$R_1^2 = 225, \quad R_0^2 = 36;$$

$$\therefore p = \frac{15 \times 36}{189} \left[ \frac{225}{r^2} - 1 \right];$$

$$p' = -\frac{15 \times 36}{189} \left[ \frac{225}{r^2} + 1 \right].$$

*Table of Stresses.*

$r$ in Inches	Tons per Square Inch	
	$p$	$p'$
6	15	20.7
7	10.3	16.0
8	7.15	12.9
9	5.15	10.9
10	3.6	9.3
11	2.47	8.15
12	1.6	7.3
13	.945	6.7
14	.315	6.02
15	0	5.72

The curves roughly plotted in Fig. 269 show that the radial and hoop stress diminish very rapidly as we pass from the inner to the

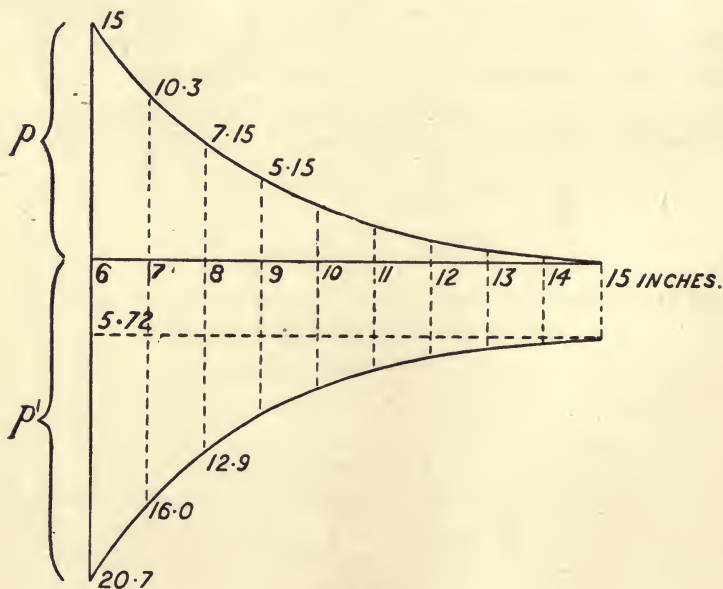


Fig. 269.

outer surface. The high value of the maximum hoop stress (20.7 tons per sq. inch) shows that the material would fail unless it had previously been put into a state of compression by shrinking on a ring.

3. *The cast-iron cylinder of a hydraulic accumulator is 9 inches internal diameter. Find the thickness required to resist an internal pressure of 800 lbs. per sq. inch. Take the maximum safe stress as 2200 lbs. per sq. inch.*

$$p' = -\frac{p_0 R_0^2}{R_1^2 - R_0^2} \left[ \frac{R_1^2}{r^2} + 1 \right],$$

at the inside

$$p' = 2200 \text{ lbs. per sq. inch ;}$$

$$r = R_0 = 4.5 \text{ inches ;}$$

therefore

$$2200 = \frac{800 (R_1^2 + (4.5)^2)}{R_1^2 - (4.5)^2} ;$$

$$R_1 = 6.35 \text{ inches ;}$$

therefore thickness

$$= 6.35 - 4.5 = 1.85 \text{ inches.}$$

### EXERCISES.

1. What should be the thickness of a 12-in. cylinder which has to sustain an internal pressure of 4000 lbs. per sq. inch ; the maximum tensile stress being limited to 10,000 lbs. per sq. inch ?

2. A hydraulic press cylinder is 12 ins. external diameter and 2 ins. thick. Internally it sustains a pressure of 2 tons per sq. inch. Find the hoop tension at the inner and outer surfaces, and graph its value throughout the thickness.

## CHAPTER XII.

### CONTINUOUS GIRDERS.

**121.** WHEN a girder is supported at more than two points it is said to be continuous. When loaded a portion of each span near the supports is bent convex upwards, the upper fibres being in tension, and the lower fibres in compression. The central portion of each span is bent concave upwards, the upper fibres being in compression, and the lower fibres in tension just as in a loaded girder supported at two points. At the points of contrary flexure, or points of inflection, the curvature changes sign, the bending moment is zero, and consequently the flange stresses are zero.

**122.** To find the bending moment at any section of a span of a continuous girder loaded with a uniform load.

Let  $l_1$  be the length of the span 1—2 (Fig. 270).

„  $w_1$  be weight of the uniform load per foot run.

„  $x$  be the abscissa of any section  $K$  referred to support 1 as origin.

„  $M_1$  and  $M_2$  be the moments of the elastic forces at the supports 1 and 2 respectively.

„  $M$  be the bending moment at  $K$ .

„  $F$  be the shearing force at  $K$ .

„  $F_1$  be the shearing force on a section in span 1—2, infinitely near to and on the right of support 1.

„  $F_2'$  the shearing force on a section in the span 1—2, infinitely near to and on the left of support 2.

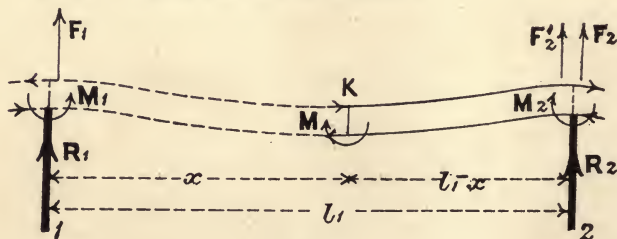


Fig. 270.

Considering the separate equilibrium of the portion  $(l_1 - x)$  of the span, we get

$$F_2' (l_1 - x) - M_2 - \frac{w_1 (l_1 - x)^2}{2} - M = 0,$$

$$\text{or} \quad M = -M_2 + F_2' (l_1 - x) - \frac{w_1 (l_1 - x)^2}{2} \dots\dots\dots(1).$$

Again, considering the whole span, we have

$$F_2' l_1 + M_1 - M_2 - \frac{w_1 l_1^2}{2} = 0,$$

$$\text{or,} \quad F_2' = \frac{M_2 - M_1}{l_1} + \frac{w_1 l_1}{2} \dots\dots\dots(2).$$

Substituting in (1) for  $F_2'$  its value from (2),

$$M = -M_2 + (M_2 - M_1) + \frac{w_1 l_1^2}{2} - (M_2 - M_1) \frac{x}{l_1} - \frac{w_1 l_1 x}{2} - \frac{w}{2} (l_1^2 - 2l_1 x + x^2)$$

$$= -M_1 - (M_2 - M_1) \frac{x}{l_1} + \frac{w_1 x}{2} (l_1 - x) \dots\dots\dots(3),$$

$$= -M_1 - (M_2 - M_1) \frac{x}{l_1} + m$$

where  $m$  is the bending moment at the section  $K$  for a span  $l_1$  similarly loaded, but merely supported at the ends.

The shearing force at  $K$  is

$$F = F_2' - w_1 (l_1 - x) \dots\dots\dots(4),$$

which for  $x = 0$ , gives by equation (2)

$$\begin{aligned} F_1 &= F_2' - w_1 l_1 \\ &= \frac{M_2 - M_1}{l_1} - \frac{w_1 l_1}{2} \dots\dots\dots(5). \end{aligned}$$

From these equations the bending moment and shearing force at any section of the span can be found, when the moments at each extremity of the span are known.

The maximum value of  $M$  at any section intermediate between the points of inflection occurs where the shearing force  $F$  changes sign, its position being got by solving for  $x$  in the equation

$$F = 0,$$

or by equations (2) and (4)

$$\frac{M_2 - M_1}{l_1} + \frac{w_1 l_1}{2} - w_1 (l_1 - x) = 0;$$

$$\text{hence,} \quad x = \frac{l_1}{2} - \frac{1}{w_1 l_1} \{M_2 - M_1\} \dots\dots\dots(6).$$

The substitution in equation (3) of the value of  $x$  obtained from (6) will give the maximum bending moment occurring between the points of inflection.

The positions of the points of inflection are got by solving for  $x$  in the equation

$$M=0;$$

$$\text{or} \quad -M_1 - (M_2 - M_1) \frac{x}{l_1} + \frac{w_1 x}{2} (l_1 - x) = 0 \dots \dots \dots (7).$$

**123. Graphic representation of the bending moment at any section of a given span.**

Let 1 and 2 be the supports of span 1—2 (Fig. 271) of length  $l_1$ . Draw  $1a$ ,  $2b$ , perpendicular to 12 to represent the moments at 1 and 2; then  $fd$  the ordinate to  $ab$  at abscissa  $x$  represents the value of the two first terms of equation (3), since

$$fd = - \left( M_1 + (M_2 - M_1) \frac{x}{l_1} \right) = -M_1 - (M_2 - M_1) \frac{x}{l_1}.$$

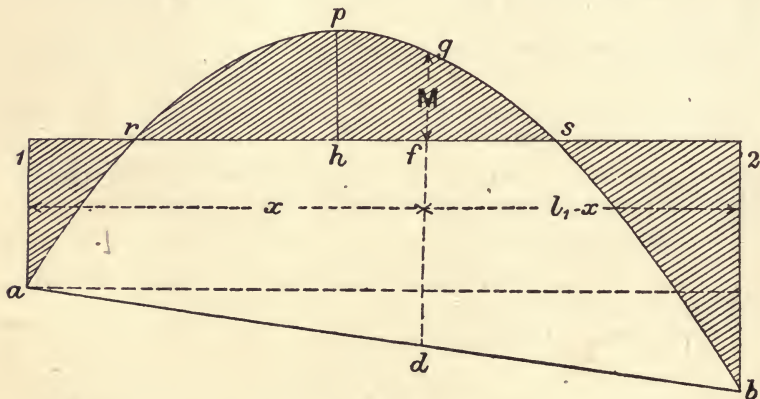


Fig. 271.

But the ordinate  $dg$ , measured upwards to the parabola  $abp$ , represents the third term of that equation

$$m = \frac{w_1 x}{2} (l_1 - x).$$

Thus the ordinate  $fg$  represents  $M$ , the bending moment at  $x$ .

To draw the parabola, find  $x = 1h$  by solving equation (6), substitute this value of  $x$  in the equation

$$M = -M_1 - (M_2 - M_1) \frac{x}{l_1} + \frac{w_1 x}{2} (l_1 - x),$$

and draw  $hp$  to represent the corresponding value of  $M$ ; then  $p$  is the vertex of the parabola.

The points of inflection are at  $r$  and  $s$  where  $M=0$ ; they can be obtained from equation (7).

The bending moment at any point is represented by the ordinate of the shaded area.

### 124. Theorem of three moments.

To determine a relation between the *bending moments at any three consecutive supports* of a uniform and uniformly loaded continuous girder resting on a number of supports, all of which are on the same level.

Let 1, 2, 3 be three consecutive supports on the same level for a continuous girder over any number of spans (Fig. 272).

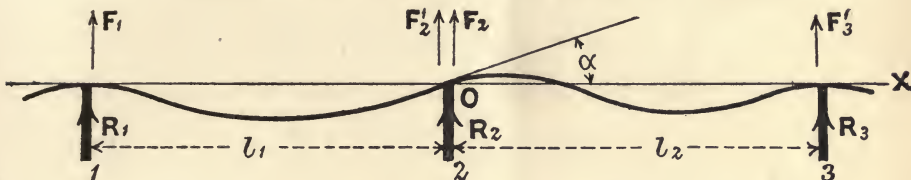


Fig. 272.

Let  $l_1$  = length of span 1—2.

„  $l_2$  = length of span 2—3.

„  $w_1, w_2$  = loads per unit of length on the spans 1—2, 2—3, respectively.

„  $R_1, R_2, R_3$  be the reactions at supports 1, 2, 3, respectively.

„  $M_1, M_2, M_3$  be the bending moments at 1, 2, 3, respectively.

„  $F_1$  be the shear on a section in span 1—2, very close to support 1.

„  $F_2'$  be the shear on a section in span 1—2, very close to support 2.

„  $F_2$  be the shear on a section in span 2—3, very close to support 2.

„  $F_3'$  be the shear on a section in span 2—3, very close to support 3.

„  $\alpha$  be the angle which the tangent to the girder at 2 makes with the horizontal.

Take  $O$  at support 2 as origin, and 2—3 as the axis of  $x$ .

Consider the span 2—3. The bending moment at any point  $(x, y)$  is

$$EI \frac{d^2y}{dx^2} = M = M_2 + F_2x - \frac{w_2x^2}{2} \dots\dots\dots(1)$$

at support 3 ;  $x = l_2$  and  $M = M_3$ .

$$\text{Therefore} \quad M_3 = M_2 + F_2l_2 - \frac{w_2l_2^2}{2} \dots\dots\dots(2).$$

Integrating equation (1),

$$EI \frac{dy}{dx} = M_2x + \frac{1}{2}F_2x^2 - \frac{w_2x^3}{6} + C$$

when  $x = 0$ ,  $\frac{dy}{dx} = \tan \alpha$  ; hence  $C = EI \tan \alpha$ .

Therefore

$$EI \left( \frac{dy}{dx} - \tan \alpha \right) = M_2 x + \frac{1}{2} F_2 x^2 - \frac{w_2 x^3}{6}.$$

Integrating again,

$$EI (y - x \tan \alpha) = \frac{1}{2} M_2 x^2 + \frac{1}{6} F_2 x^3 - \frac{w_2 x^4}{24}.$$

There is no constant of integration, for when  $x = 0$ ;  $y = 0$ .

Again, when  $x = l_2$ ;  $y = 0$ , hence,

$$-EI \tan \alpha = \frac{1}{2} M_2 l_2 + \frac{1}{6} F_2 l_2^3 - \frac{w_2 l_2^4}{24}.$$

Substituting for  $F_2$  its value from (2), we get

$$-EI \tan \alpha = \frac{M_3 l_2}{6} + \frac{M_2 l_2}{3} + \frac{w_2 l_2^3}{24} \dots \dots \dots (3).$$

Similarly for the span 1—2, we get by substituting  $-\tan \alpha$  for  $\tan \alpha$ ,

$$EI \tan \alpha = \frac{M_1 l_1}{6} + \frac{M_2 l_1}{3} + \frac{w_1 l_1^3}{24} \dots \dots \dots (4).$$

Hence by adding (3) and (4),

$$(M_1 + 2M_2) l_1 + (M_3 + 2M_2) l_2 + \frac{1}{4} (w_1 l_1^3 + w_2 l_2^3) = 0 \dots \dots (5).$$

This relation is called the *theorem of the three moments*. If there are  $n$  supports, we get  $n - 2$  equations connecting the corresponding bending moments, and two other equations are given by the conditions of support at the ends. Thus if the girder is merely supported at the ends,  $M_1 = 0$  and  $M_n = 0$ ; if an end is fixed  $\frac{dy}{dx} = 0$  at that support.

From the values of  $M_1, M_2, \dots, M_n$  thus obtained we can determine the bending moment at any section of a given span.

The section of maximum bending moment is got by making  $\frac{dM}{dx} = 0$ ; and the points of inflection by solution of the equation  $M = 0$ .

Thus considering span 2—3,

$$M = M_2 + F_2 x - \frac{w_2 x^2}{2},$$

$$\frac{dM}{dx} = F_2 - w_2 x = 0.$$

Therefore

$$x = \frac{F_2}{w_2},$$

and max. bending moment  $= M_2 + \frac{F_2^2}{2w_2}.$

For concentrated loads, the theorem of three moments becomes

$$(M_1 + 2M_2) l_1 + (M_3 + 2M_2) l_2 + \Sigma \frac{W_1 x_1}{l_1} (l_1^2 - x_1^2) + \Sigma \frac{W_2 x_2}{l_2} (l_2^2 - x_2^2) = 0,$$

where  $W_1$  and  $W_2$  are the loads in the two spans  $l_1$  and  $l_2$  respectively, and  $x_1$  and  $x_2$  the distances of those loads to the supports at the extremities of the span under consideration. If there are a number of loads in one span take the algebraic sum of the moments, *i.e.*  $\Sigma Wx$ .

### 125. Reactions.

The reaction at any support is the sum of the shearing forces on each side of that support. This is evident if we consider the separate equilibrium of the *very small* portion of the girder between the sections on which  $F_2$  and  $F_2'$  act; the reaction  $R_2$  which is equal and opposite to the pressure on the support, must for equilibrium be equal to the sum of the shearing forces, thus

$$R_2 = F_2 + F_2' \dots\dots\dots(6),$$

and at any support  $n$   $R_n = F_n + F_n' \dots\dots\dots(7).$

At the two extreme ends, where the girder is merely supported, the reaction is equal to the shearing force.

*To find the reaction  $R_2$  at support 2 (Fig. 272).*

Consider the equilibrium of the span 2—3. Taking moments about support 3, we get

$$M_3 = F_2 l_2 - \frac{w_2 l_2^2}{2} + M_2,$$

or  $F_2 = \frac{M_3 - M_2}{l_2} + \frac{w_2 l_2}{2} \dots\dots\dots(8).$

Again, considering the span 1—2,

$$M_1 = F_2' l_1 - \frac{w_1 l_1^2}{2} + M_2,$$

or  $F_2' = \frac{M_1 - M_2}{l_1} + \frac{w_1 l_1}{2} \dots\dots\dots(9).$

Therefore, by adding (8) and (9),

$$R_2 = \frac{M_1 - M_2}{l_1} + \frac{M_3 - M_2}{l_2} + \frac{w_1 l_1}{2} + \frac{w_2 l_2}{2} \dots\dots\dots(10);$$

and generally at any intermediate support  $n$  separating the spans  $l_{n-1}$  and  $l_n$ ,

$$R_n = \frac{M_{n-1} - M_n}{l_{n-1}} + \frac{M_{n+1} - M_n}{l_n} + \frac{w_{n-1} l_{n-1}}{2} + \frac{w_n l_n}{2} \dots\dots\dots(11).$$

If there are  $r$  supports, 1, 2, 3, ...,  $r$ , with spans  $l_1, l_2, \dots, l_{r-1}$ , and the girder is free over the supports 1 and  $r$ , then evidently

$$R_1 = F_1 = \frac{M_2}{l_1} + \frac{w_1 l_1}{2} \dots\dots\dots(12)$$

and  $R_r = F_r' = \frac{M_{r-1}}{l_{r-1}} + \frac{w_{r-1} l_{r-1}}{2} \dots\dots\dots(13).$

Thus, having found the values of the moments at each support by the equation of the theory of three moments, the reactions can be at once obtained.

*Examples.*

1. Find the bending moment at the middle support of a continuous girder of two unequal spans, the left one of length 40 feet carrying 2 tons per foot run, and the right one of length 30 feet carrying 1 ton per foot run. Find also the reactions at each support.

The equation of three moments (5) is

$$(M_1 + 2M_2)l_1 + (M_3 + 2M_2)l_2 = -\frac{1}{4}(w_1l_1^3 + w_2l_2^3),$$

but as  $M_1 = 0$ , and  $M_3 = 0$ , we get

$$8M_2(l_1 + l_2) = -w_1l_1^3 - w_2l_2^3,$$

$$560M_2 = -2 \times 40^3 - 1 \times 30^3 = -91000,$$

$$M_2 = -162.5 \text{ foot-tons.}$$

By equation (12), or from moments about support 2,

$$R_1 = \frac{-162.5}{40} + \frac{2 \times 40}{2} = 35.94 \text{ tons.}$$

By equations (10) or (11), putting  $M_1 = 0$ ,  $M_3 = 0$ ,  $M_2 = -162.5$ ;

$$R_2 = \frac{162.5}{40} + \frac{162.5}{30} + \frac{2 \times 40}{2} + \frac{1 \times 30}{2} \\ = 64.48 \text{ tons.}$$

By equation (13), or by moments round support 2,

$$R_3 = \frac{-162.5}{30} + \frac{1 \times 30}{2} \\ = 9.58 \text{ tons.}$$

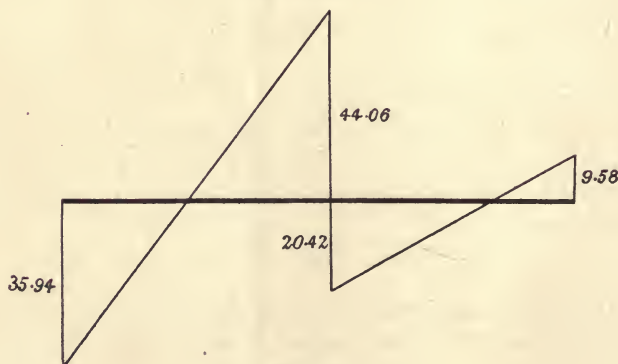


Fig. 273.

The results can be verified thus :

$$R_1 + R_2 + R_3 - w_1l_1 - w_2l_2 = 0,$$

$$35.94 + 64.48 + 9.58 - 80 - 30 = 0,$$

or

$$110 - 110 = 0,$$

which shows that the values of  $R_1$ ,  $R_2$ ,  $R_3$  are correct.

The shearing force diagram is sketched in Fig. 273, and the bending moment diagram in Fig. 274.

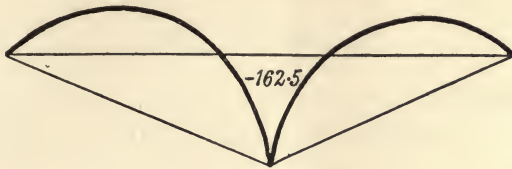


Fig. 274.

2. A continuous girder of three spans has two equal end spans of 240 feet, and a centre span of 150 feet; the supports are level and the girders are free over the abutment piers, and are assumed to be of uniform section. The fixed load carried by each girder is  $\frac{1}{2}$  ton per foot, and the moving load is 1 ton per foot. Calculate the bending moments over the two central supports when the left end span only is covered by the moving load, and then determine the maximum positive bending moment occurring on a section in that span. Find also the reactions at each support.

Adopting the notation of previous articles we have

$$\begin{aligned} l_1 &= 240 \text{ feet, } l_2 = 150 \text{ feet, } l_3 = 240 \text{ feet,} \\ w_1 &= 1\frac{1}{2} \text{ ton per foot, } w_2 = w_3 = \frac{1}{2} \text{ ton per foot,} \\ M_1 &= M_4 = 0. \end{aligned}$$

Equation of three moments for spans 1—2, 2—3, is

$$(M_1 + 2M_2) l_1 + (M_3 + 2M_2) l_2 = - \frac{(w_1 l_1^3 + w_2 l_2^3)}{4}.$$

Substituting the numerical values above, we get

$$\begin{aligned} 8M_2(240 + 150) + 4M_3 \times 150 &= -\frac{3}{2} \times 240^3 - \frac{1}{2} \times 150^3, \\ \text{or, } 5 \cdot 2M_2 + M_3 &= -37372 \cdot 5 \dots\dots\dots (A). \end{aligned}$$

For spans 2—3, 3—4, similarly

$$5 \cdot 2M_3 + M_2 = -14332 \cdot 5 \dots\dots\dots (B).$$

Solving for  $M_2$  from equations (A) and (B), we get

$$\begin{aligned} 26 \cdot 04M_2 &= -180004 \cdot 5, \\ M_2 &= -6912 \cdot 6 \text{ foot-tons,} \\ M_3 &= -37372 \cdot 5 - 5 \cdot 2 \times -6912 \cdot 6 \\ &= -1427 \text{ foot-tons.} \end{aligned}$$

Maximum positive bending moment, span 1—2.

At any section distant  $x$  from support 1, the bending moment is

$$M = M_1 + F_1 x - \frac{wx^2}{2} \dots\dots\dots (C).$$

At support 2, where  $x = l_1$ ,  $M = M_2$ , and

$$M_2 = M_1 + F_1 l_1 - \frac{w_1 l_1^2}{2},$$

or

$$F_1 = \frac{M_2 - M_1}{l_1} + \frac{w_1 l_1}{2} \dots \dots \dots (D).$$

Substituting in (C),

$$M = M_1 + (M_2 - M_1) \frac{x}{l_1} + \frac{w_1 x}{2} (l_1 - x) \dots \dots \dots (E);$$

but since  $M_1 = 0$ ,

$$M = M_2 \frac{x}{l_1} + \frac{w_1 x}{2} (l_1 - x) \\ = \frac{3}{4} x (240 - x) - \frac{x}{240} \times 6912 \cdot 6;$$

this will have its maximum positive value when  $\frac{dM}{dx} = 0$ , that is for

$$180 - \frac{2}{3} x - 28 \cdot 8 = 0,$$

or

$$x = 100 \cdot 8 \text{ feet from support 1.}$$

The *maximum* value of  $M$  required is

$$M = \frac{3}{4} \times 100 \cdot 8 \times 139 \cdot 2 - 100 \cdot 8 \times 28 \cdot 8 \\ = + 7620 \cdot 5 \text{ foot-tons.}$$

*Reactions.* Using the notation of Art. 124, we have  $M_1 = M_4 = 0$ .

Taking moments about support 2,

$$R_1 = F_1 = \frac{M_2 - M_1}{l_1} + \frac{w_1 l_1}{2} = \frac{-6912}{240} + \frac{3}{4} \times 240 \\ = \underline{151 \cdot 2 \text{ tons.}}$$

Taking moments about support 1,

$$F_2' = \frac{M_1 - M_2}{l_1} + \frac{w_1 l_1}{2} = \frac{+6912}{240} + \frac{3}{4} \times 240 = 208 \cdot 8 \text{ tons.}$$

Taking moments about support 3,

$$F_2 = \frac{M_3 - M_2}{l_2} + \frac{w_2 l_2}{2} = \frac{+5485}{150} + \frac{1}{4} \times 150 = 73 \cdot 4 \text{ tons,}$$

and

$$R_2 = F_2 + F_2' = 73 \cdot 5 + 208 \cdot 7 = \underline{282 \cdot 2 \text{ tons.}}$$

Taking moments about support 2,

$$F_3' = \frac{M_2 - M_3}{l_2} + \frac{w_2 l_2}{2} = \frac{-5485}{150} + \frac{1}{4} \times 150 = 1 \cdot 6 \text{ tons.}$$

Taking moments about support 4,

$$F_3 = \frac{M_4 - M_3}{l_3} + \frac{w_3 l_3}{2} = \frac{+1427}{240} + \frac{1}{4} \times 240 = 65 \cdot 8 \text{ tons,}$$

and

$$R_3 = F_3 + F_3' = 65 \cdot 8 + 1 \cdot 7 = \underline{67 \cdot 5 \text{ tons.}}$$

Taking moments about support 3,

$$R_4 = F_4' = \frac{M_3 - M_4}{l_3} + \frac{w_3 l_3}{2} = \frac{-1427}{240} + \frac{1}{4} \times 240 = -5 \cdot 9 + 60 = \underline{54 \cdot 2 \text{ tons.}}$$

To verify the accuracy,

$$R_1 + R_2 + R_3 + R_4 = 151.2 + 282.2 + 67.4 + 54.2 = 555,$$

which must be equal to

$$w_1 l_1 + w_2 l_2 + w_3 l_3 = 360 + 75 + 120 = 555.$$

Again, the sum of the shears at the supports *in any one span* should be equal to the total load on that span. Thus for span 2—3 on which the total load is  $w_2 l_2 = 75$  tons, we have

$$F_2 + F_3' = 73.4 + 1.6 = 75 \text{ tons.}$$

**126. Spans of equal length, each loaded with the same uniform load  $w$  per foot run.**

If  $l$  be the length of each span, the equation of three moments becomes

$$M_1 + M_3 + 4M_2 = -\frac{wl^2}{2}.$$

CASE 1. *Two equal spans.*

$$M_1 = 0 \text{ and } M_3 = 0.$$

Therefore 
$$M_2 = -\frac{wl^2}{8}.$$

Again, 
$$R_1 = F_1 = \frac{M_2}{l} + \frac{wl}{2} = \frac{wl}{2} - \frac{wl}{8} = \frac{3}{8}wl.$$

Similarly 
$$R_3 = \frac{3}{8}wl,$$
  

$$R_2 = 2wl - \frac{3}{8}wl - \frac{3}{8}wl = \frac{5}{4}wl.$$

CASE 2. *Three equal spans.*

$$M_1 = 0 \text{ and } M_4 = 0,$$

$$M_1 + M_3 + 4M_2 = -\frac{wl^2}{2}.$$

Therefore 
$$M_3 + 4M_2 = -\frac{wl^2}{2},$$

and 
$$M_2 + 4M_3 = -\frac{wl^2}{2}.$$

Solving we get 
$$M_2 = M_3 = -\frac{wl^2}{10}.$$

Again, 
$$R_1 = R_4 = -\frac{wl}{10} + \frac{wl}{2} = \frac{4}{10}wl.$$

From symmetry, and knowing that  $R_1 + R_2 + R_3 + R_4 = 3wl$ ,

$$R_2 = R_3 = \frac{1}{2} (3wl - \frac{4}{5}wl) \\ = \frac{11}{10}wl.$$

CASE 3. *Four equal spans.*

$$M_1 = 0, M_5 = 0, \text{ and from symmetry } M_2 = M_4.$$

General equation is

$$M_1 + M_3 + 4M_2 = -\frac{wl^2}{2}.$$

Therefore,  $M_3 + 4M_2 = -\frac{wl^2}{2}$  .....(A).

Again,  $M_2 + M_4 + 4M_3 = -\frac{wl^2}{2},$

or  $M_2 + 2M_3 = -\frac{wl^2}{4}$  .....(B).

From (A) and (B) we get

$$7M_3 = -\frac{wl^2}{2}.$$

Therefore  $M_3 = -\frac{wl^2}{14},$

and  $M_2 = M_4 = -\frac{3}{28}wl^2.$

*Reactions.*

$$R_1 = F_1 = \frac{M_2}{l} + \frac{wl}{2} = -\frac{3}{28}wl + \frac{wl}{2} \\ = \frac{11}{28}wl,$$

and  $R_5 = \frac{11}{28}wl.$

Moments round 1,

$$F'_2 = \frac{wl}{2} - \frac{M_2}{l} = \frac{wl}{2} + \frac{3}{28}wl = \frac{17}{28}wl.$$

Moments round 3,

$$F_2 = \frac{M_3 - M_2}{l} + \frac{wl}{2} = wl \left( -\frac{1}{14} + \frac{3}{28} + \frac{1}{2} \right) = \frac{15}{28}wl.$$

Therefore  $R_2 = F'_2 + F_2 = \frac{32}{28}wl,$

and  $R_4 = \frac{32}{28}wl.$

Again, taking moments about 2,

$$F'_3 = \frac{M_2 - M_3}{l} + \frac{wl}{2} = wl \left( -\frac{3}{28} + \frac{1}{14} + \frac{1}{2} \right) = \frac{13}{28}wl.$$

Taking moments about 4,

$$F_3 = \frac{M_4 - M_3}{l} + \frac{wl}{2} = wl \left( -\frac{3}{28} + \frac{1}{14} + \frac{1}{2} \right) = \frac{13}{28}wl.$$

Therefore  $R_3 = F'_3 + F_3 = \frac{26}{28}wl.$

## 127. Case of middle support lower than the end supports.

A girder continuous over two spans  $l_1$  and  $l_2$  rests freely on two end supports which are on the same level. If the intermediate support settle  $y_0$  inches, find how much the bending moment at this support will be diminished.

Let  $l_1, l_2$  = lengths of the two spans (Fig. 275).

„  $w_1, w_2$  = loads per foot run on same.

„  $M_2$  = bending moment at support 2.

„  $\alpha$  = slope of tangent to girder at 2.

„  $y_0$  = settlement of support 2.

„  $R_2$  = reaction at support 2.

Take the level line of supports 1 and 3 as the axis of  $x$ , and its intersection  $O$  with centre line of support 2 as origin.

Consider the span  $l_2$ , on right side of  $O$ .

Bending moment at any section ( $x$ ) is

$$M_x = M_2 + R_2 x - w_2 \frac{x^2}{2} \dots \dots \dots (1).$$

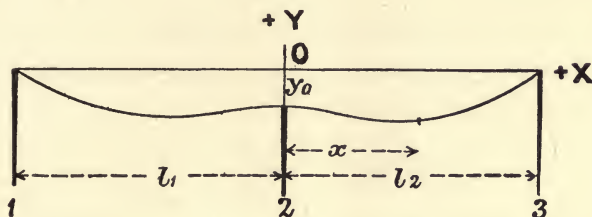


Fig. 275.

For  $x = l_2, M_x = 0$ .

Therefore  $R_2 = \frac{w_2 l_2}{2} - \frac{M_2}{l_2}$ .

Substituting for  $R_2$  in (1),

$$M_x = M_2 \left(1 - \frac{x}{l_2}\right) + \frac{w_2}{2} x (l_2 - x).$$

Therefore

$$EI \frac{d^2 y}{dx^2} = M_2 \left(1 - \frac{x}{l_2}\right) + \frac{w_2}{2} x (l_2 - x).$$

Integrating,

$$EI \left(\frac{dy}{dx} - \tan \alpha\right) = M_2 \left(x - \frac{x^2}{2l_2}\right) + \frac{w_2}{2} \left(\frac{l_2 x^2}{2} - \frac{x^3}{3}\right).$$

Integrating again,

$$EI (y - y_0 - x \tan \alpha) = M_2 \left(\frac{x^2}{2} - \frac{x^3}{6l_2}\right) + \frac{w_2}{2} \left(\frac{l_2 x^3}{6} - \frac{x^4}{12}\right).$$

For  $x = l_2, y = 0$ , therefore

$$EI (y_0 + l_2 \tan \alpha) + M_2 \frac{l_2^2}{3} + \frac{w_2 l_2^4}{24} = 0,$$

or

$$2M_2 l_2^2 + \frac{w_2 l_2^4}{4} + 6EI y_0 + 6EI l_2 \tan \alpha = 0 \dots \dots \dots (A).$$

Similarly for the span  $l_1$ , using  $-\tan \alpha$  instead of  $\tan \alpha$ , we have

$$2M_2 l_1^2 + \frac{w_1 l_1^4}{4} + 6EI y_0 - 6EI l_1 \tan \alpha = 0 \quad \dots\dots\dots (B).$$

Multiply equation (A) by  $l_1$ , equation (B) by  $l_2$ , and add,

$$2M_2 (l_1 + l_2) l_1 l_2 + \frac{1}{4} (w_1 l_1^3 + w_2 l_2^3) l_1 l_2 = -6EI (l_1 + l_2) y_0,$$

or 
$$M_2 = -\frac{1}{8} \frac{w_1 l_1^3 + w_2 l_2^3}{l_1 + l_2} - \frac{3y_0}{l_1 l_2} EI \dots\dots\dots (C).$$

When  $y_0 = 0$ , equation (C) is the form which the equation expressing the "Theorem of Three Moments" reduces to, for the special case of two spans with the three supports on the same level.

#### *Example.*

Let  $l_1 = l_2 = 100 \text{ ft.} = 1200 \text{ ins.}$

Let  $y_0 =$  the settlement of the central support  $= -1 \text{ inch.}$

Then the last term of equation (C) is

$$+ \frac{3EI}{1200 \times 1200} = + \frac{EI}{480000} \text{ inch-lbs.}$$

and expresses the diminution in the amount of the negative bending moment at support 2 due to the sinking of the centre support.

### 128. Advantages and disadvantages of continuous girders

In the case of separate spans the bending moment is greatest near the centre, whereas in the continuous girder the maximum bending moments occur near the supports, also the average value of the bending moment is less, thus there is a saving in the flange material, and the heavier sections are placed over the supports, which means that a portion of the weight is removed from the centre towards the supports.

The disadvantages of continuity are chiefly due to the effect of rolling loads which alter the positions of the points of inflection, and portions of the span are subjected to bending moments which change in sign and amount, the members there being exposed to stresses which are alternately tensile and compressive, especially so when dead load on the bridge is light as compared to the live load. Another disadvantage of continuous girders is that settlement in the supports also causes the points of inflection to change, and may considerably alter the stresses calculated on the assumption that all the supports are level. The Moment of Inertia  $I$  is not constant; it is subject to variation.

## EXERCISES.

1. A continuous girder covers 2 spans of unequal length, the left one of span 30 feet carrying 3 tons per foot run, the other of 20 feet carrying 1 ton per foot run. Find the bending moment at the middle support, and the reaction at each support.

$$\text{Ans. } M_2 = -222.5 \text{ foot-tons.}$$

$$R_1 = 37.58 \text{ tons.}$$

$$R_2 = 73.54 \text{ tons.}$$

$$R_3 = -1.12 \text{ tons.}$$

2. A. A continuous girder of uniform section rests on three level supports. The spans are 100 feet and 150 feet. The girder carries a dead load of 0.4 ton per foot, and a moving load of 0.6 ton per foot. Calculate the bending moment at the intermediate support, and the reactions at the three supports, when both spans are covered by the moving load.

B. If the longer span only is covered with the moving load, determine the bending moment at the middle support, and on a section in the centre of the longer span.

$$\text{Ans. A. } -2187.5 \text{ foot-tons ; } 28.13 \text{ tons ; } 161.45 \text{ tons ; } 60.42 \text{ tons.}$$

$$\text{B. } M_2 = -1887.5 \text{ foot-tons ; } M = 1868.75 \text{ foot-tons.}$$

3. A continuous girder covers three equal spans of 40 feet, each carrying a uniform load of 2 tons per foot. Determine the bending moments at the two intermediate supports, the position and value of the maximum positive bending moments in each span, and the points of inflection for each span.

$$\text{Ans. } M_2 = M_3 = -320 \text{ foot-tons,}$$

$$\text{at } x = 16 \text{ feet right of 1, } M = +256 \text{ foot-tons,}$$

$$\text{at } x = 20 \text{ feet right of 2, } M = +80 \text{ foot-tons,}$$

$$\text{at } x = 24 \text{ feet right of 3, } M = +256 \text{ foot-tons.}$$

Points of inflection :

Span 1—2 ; 32 feet from 1.

„ 2—3 ; 11.1 feet and 28.9 feet from 2.

„ 3—4 ; 8 feet from 3.

4. A girder is continuous over three spans, the two end spans  $l_1$  and  $l_3$  each 50 feet long, the central span  $l_2$  70 feet long. It rests freely on its end supports, and carries a dead load of 1 ton per foot run over its whole length, and a live load of  $1\frac{1}{2}$  tons per foot.

When the moving load covers spans  $l_1$  and  $l_2$ , determine the

bending moments at supports 2 and 3, the reaction at support 2, and the position of the points of inflection in span 2—3 ( $l_2$ ).

*Ans.*  $M_2 = -1005.8$  foot-tons;  $M_3 = -730.1$  foot-tons;  
 $R_2 = 174.06$  tons;  $x = 13.49$  feet and  $59.66$  feet from 2.

5. A continuous girder rests on three level supports; it is free at the end supports, and is divided by the intermediate support into two spans of 80 feet and 120 feet. It carries a uniform dead load of 0.4 ton per foot run, and a moving load of 0.6 ton per foot. (A) Calculate the bending moment at the intermediate support, and the reactions at the three supports when the moving load covers both spans. (B) Calculate the bending moment at the intermediate support, and on a section in the middle of the longer span when that span alone is covered by the moving load.

*Ans.* (A)  $M_2 = -1400$  foot-tons;  $R_1 = 22.5$  tons;  
 $R_2 = 129.17$  tons;  $R_3 = 48.33$  tons.  
 (B)  $M_2 = -1208$  foot-tons;  $R_2 = 49.93$  tons;  
 $M = +1196$  foot-tons.

6. A continuous girder covering two equal spans of 50 feet carries a load of 10 tons at a distance of  $12\frac{1}{2}$  feet from the left support. Calculate the bending moment at the central pier, and the reactions at each pier.

*Ans.*  $M_2 = -29.3$  foot-tons;  
 $R_1 = 6.92$  tons;  $R_2 = 3.66$  tons;  $R_3 = -0.58$  ton.

7. A girder continuous over two equal spans supports a load of 2000 lbs. at the centre of first span, and a load of 1000 lbs. at the centre of the second span. Find the reactions.

*Ans.*  $R_1 = 718.75$  lbs.;  $R_2 = 2062.50$  lbs.  
 $R_3 = 218.75$  lbs.

8. A girder of uniform section, loaded with a uniform load, is continuous over two equal spans. What is the gain in strength and stiffness as compared with two discontinuous girders of the same section similarly loaded?

*Ans.* Strength the same. Stiffness in the ratio of 5 to 2.

## CHAPTER XIII.

### CANTILEVER BRIDGES.—SUSPENSION BRIDGES.—ARCHED RIBS.

129. THE disadvantages of continuous girders are removed if hinges are introduced at the points of contrary flexure. The bridge is then composed of cantilevers and suspended girders; and there is no ambiguity regarding the stresses. The advantage of the continuous girder is preserved, and its chief disadvantage is avoided.

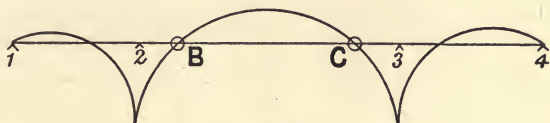


Fig. 276.

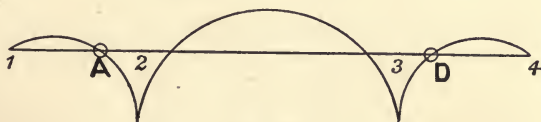


Fig. 277.

In Fig. 276, let 1, 2, 3, 4 be the points of support :

(a) The hinges may be introduced in the central span at *B* and *C*, then these points become the points of contrary flexure; and the portion *BC* may be treated as an independent girder supported at the ends by the cantilever arms *2B* and *3C*. In this case the side spans must be anchored down at 1 and 4, as the reactions at these points may become negative, that is, the girders may exert a lifting force.

The line *BC* becomes the datum line for the bending moment diagram.

(b) The hinges may be introduced in the side spans at *A* and *D* (Fig. 277). In this case the reactions at 1 and 4 are always positive, as the girders cannot exert any lifting force at these points.

*Note.* The hinges may be placed in the central span *or* in the side spans, but *not* in both.

**130. CASE A. Hinges in the central span (Fig. 278).***I. Uniform load of intensity  $w$  per foot run.*

Let  $R_1, R_2, R_3, R_4$  be the reactions at the points of support 1, 2, 3, 4 respectively.

Assume the spans symmetrical.

Let  $l_1$  be the length of each of the two side spans.

„  $a$  be the distance from support 2 to the first hinge  $B$ .

„  $b$  be the distance between the hinges.

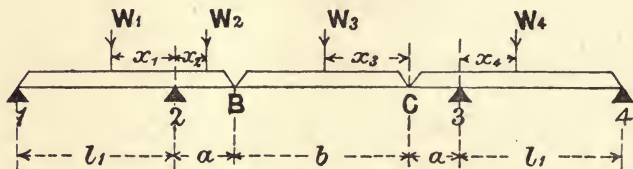


Fig. 278.

Now treat the portion  $BC$  as an independent girder supported at the ends. The stresses in it are those due to its own loads only.  $1B$  and  $4C$  can also be treated as independent girders loaded with their own loads, *and* the weights at the ends  $B$  and  $C$ , equal to the reactions at these points due to the load on the girder  $BC$ .

The bending moments at piers 2 and 3 are

$$M_2 = M_3 = -\frac{wa^2}{2} - \frac{wb}{2}a = -\frac{w}{2}(a^2 + ab).$$

To find  $R_1$ , take moments about pier 2,

$$R_1 l_1 - \frac{wl_1^2}{2} + \frac{wa^2}{2} + \frac{wb}{2}a = 0,$$

$$R_1 = \frac{wl_1}{2} - \frac{w}{2} \left( \frac{a^2 + ab}{l_1} \right).$$

To find  $R_2$ , take moments about pier 1,

$$-R_2 l_1 + \frac{wl_1^2}{2} + wa \left( l_1 + \frac{a}{2} \right) + \frac{wb}{2}(l_1 + a),$$

$$R_2 = \frac{w}{2}(l_1 + 2a + b) + \frac{w}{2} \left( \frac{a^2 + ab}{l_1} \right).$$

$R_1 = R_4$  and  $R_2 = R_3$  from symmetry.

*II. Concentrated loads.*

A load  $W_1$  on the side span 1—2 distant  $x_1$  from pier 2.

A load  $W_2$  on the cantilever arm  $2B$  distant  $x_2$  from pier 2.

A load  $W_3$  on the suspended girder distant  $x_3$  from hinge  $C$ .

A load  $W_4$  on the side span 3—4 distant  $x_4$  from pier 3.

$$\text{Load at } B = \frac{W_3 x_3}{b}; \quad \text{Load at } C = W_3 \left( \frac{b - x_3}{b} \right).$$

Bending moment at pier 2,

$$M_2 = W_2 x_2 + W_3 x_3 \frac{a}{b}.$$

Bending moment at pier 3,

$$M_3 = W_3 \left( \frac{b - x_3}{b} \right) a.$$

To find  $R_1$ , take moments about 2,

$$R_1 l_1 = W_1 x_1 - W_2 x_2 - \frac{W_3 x_3}{b} a = W_1 x_1 - M_2.$$

To find  $R_4$ , take moments about 3,

$$R_4 l_1 = W_4 x_4 - W_3 \left( \frac{b - x_3}{b} \right) a = W_4 x_4 - M_3.$$

To find  $R_2$ , take moments about 1,

$$R_2 l_1 = W_1 (l_1 - x_1) + W_2 (l_1 + x_2) + \frac{W_3 x_3}{b} (l_1 + a).$$

To find  $R_3$ , take moments about 4,

$$R_3 l_1 = W_3 \left( \frac{b - x_3}{b} \right) (l_1 + a) - W_4 (l_1 - x_4).$$

We see in both the cases considered, that

$R_1$  and  $R_4$  may be negative,

$R_2$  and  $R_3$  are always positive.

The pier moments are determined solely by the loads on the span containing the hinges, *i.e.* the central span.

### 131. CASE B. Bridge hinged in the side spans.

Let  $B$  and  $C$  be the hinges in this case (Fig. 279).

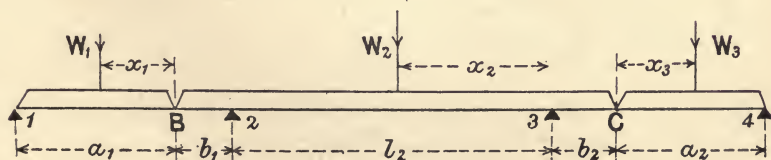


Fig. 279.

The portions  $1B$  and  $4C$  may be considered as independent girders supported at the ends, and the part  $BC$  as an independent girder supported at 2 and 3, carrying its own loads, and in addition the weights at  $B$  and  $C$  equal to the reactions at these points due to the loads on  $1B$  and  $4C$ .

Let  $\overline{1B} = a_1$ ;  $\overline{B2} = b_1$ ;  $\overline{2-3} = l_2$ ;  $\overline{3C} = b_2$ ;  $\overline{C4} = a_2$ .

(1) *A uniform load of intensity  $w$  per foot run.*

Taking 1B and 4C as independent girders,

$$R_1 a_1 = \frac{w a_1^2}{2}. \quad \therefore R_1 = \frac{w a_1}{2}.$$

$$R_4 a_2 = \frac{w a_2^2}{2}. \quad \therefore R_4 = \frac{w a_2}{2}.$$

$R_1$  and  $R_4$  are always positive; there can be no lifting force at 1 or 4, consequently no anchorage will be needed at these points.

Taking BC as an independent girder,

$$\text{Load at } B = w a_1 - R_1 = \frac{w a_1}{2};$$

$$\text{Load at } C = \frac{w a_2}{2}.$$

Take moments about pier 3,

$$R_2 l_2 = \frac{w a_1}{2} (b + l_2) + w b_1 \left( \frac{b_1}{2} + l_2 \right) + \frac{w l_2^2}{2} - \frac{w b_2^2}{2} - \frac{w a_2}{2} b_2 = 0.$$

If  $a_1 = a_2$ , and  $b_1 = b_2$ ,

$$R_2 = \frac{w}{2} (a_1 + 2b_1 + l_2).$$

The bending moment at pier 2,

$$M_2 = -\frac{w a_1}{2} b_1 - \frac{w b_1^2}{2}.$$

Bending moment at any section distant  $x$  from pier 2,

$$M_x = R_2 x - \frac{w a_1}{2} (b_1 + x) - w b_1 \left( \frac{b_1}{2} + x \right) - \frac{w x^2}{2}.$$

(2) *Concentrated loads.*

A load  $W_1$  on 1B distant  $x_1$  from B.

A load  $W_2$  on the span 2—3 distant  $x_2$  from 3.

A load  $W_3$  on 4C distant  $x_3$  from C.

In this case, we get the reactions

$$R_1 = \frac{W_1 x_1}{a_1},$$

$$R_4 = \frac{W_3 x_3}{a_2}.$$

$$\text{Load at } B = W_1 - R_1 = W_1 \left( 1 - \frac{x_1}{a_1} \right);$$

$$\text{Load at } C = W_3 - R_4 = W_3 \left( 1 - \frac{x_3}{a_2} \right).$$

Taking moments about pier 3,

$$R_2 l_2 = W_1 \left( 1 - \frac{x_1}{a_1} \right) (b_1 + l_2) + W_2 x_2 - W_3 \left( 1 - \frac{x_3}{a_2} \right) b_2.$$

Taking moments about pier 2,

$$R_3 l_2 = W_3 \left(1 - \frac{x_3}{a_2}\right) (b_2 + l_2) + W_2 (l_2 - x_2) - W_1 \left(1 - \frac{x_1}{a_1}\right) b_1.$$

These equations give  $R_2$  and  $R_3$ .

The bending moments at piers 2 and 3 are

$$M_2 = -W_1 \left(1 - \frac{x_1}{a_1}\right) b_1,$$

$$M_3 = -W_3 \left(1 - \frac{x_3}{a_2}\right) b_2.$$

Here again we see that the moments at the piers are determined solely from the loads on the spans containing the hinges.

### 132. Suspension Bridges.

In a suspension bridge, the platform is suspended by steel rods from link or wire rope cables, which pass over towers built on piers, and are securely anchored down at the ends.

When a chain of uniform weight per foot of length is suspended and hangs freely it takes the form of a *catenary* curve.

In practice, however, the loads are usually suspended from the cables by rods placed at equal distances apart, and the load is assumed to be uniform per horizontal foot run of span. The curve of the cable or chain is then a *parabola*.

### 133. Chain uniformly loaded per foot run of span.

Let  $AOB$ , Fig. 280, be the chain suspended at  $A$  and  $B$ .

„  $w$  = uniform load per foot run of span.

„  $l$  = length of span.

„  $d$  = dip, or depth of lowest point of curve below horizontal  $AB$ .

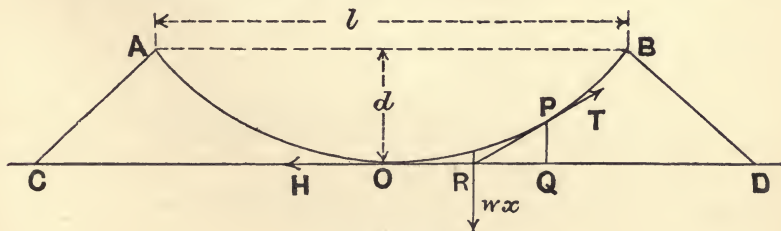


Fig. 280.

Take  $O$  the lowest point of chain as origin.

Let  $x, y$ , be the coordinates of any point  $P$  of the chain.

The portion  $OP$  of the chain is kept in equilibrium by:

- (1) The weight  $wx$ , acting at  $R$ , the middle point of  $OQ$ .
- (2) The tension at  $P$ , acting tangentially to the chain.
- (3) The horizontal tension  $H$  at  $O$ .

These three forces must meet in the same point  $R$ , and  $PQR$  is a triangle of forces.

Therefore 
$$\frac{y}{x} = \frac{wx}{H},$$

or 
$$y = \frac{wx^2}{2H} \dots \dots \dots (1),$$

which is the equation to a *parabola* with its vertex at  $O$ .

From (1) 
$$H = \frac{wx^2}{2y} \dots \dots \dots (2).$$

Let  $T$  = tension at  $P$ , then

$$T^2 = (wx)^2 + H^2 = w^2x^2 + \frac{w^2x^4}{4y^2}.$$

Therefore 
$$T = wx \sqrt{1 + \frac{x^2}{4y^2}} \dots \dots \dots (3).$$

This equation gives the tension at any point of the chain. At the ends  $A$  and  $B$ , where  $x = \frac{l}{2}$ ;  $y = d$ : we get from equations (2) and (3)

$$H = \frac{wl^2}{8d},$$

$$T = \frac{wl}{2} \sqrt{1 + \frac{l^2}{16d^2}} = \frac{wl}{8d} \sqrt{16d^2 + l^2}.$$

### 134. Pressure on the piers.

In Fig. 280,  $AOB$  is the main chain or cable,  $AC$  and  $BD$  are the side chains or backstays which are anchored down at  $C$  and  $D$ . There are two methods of carrying the chain over the piers.

(a) The main chain and backstays may be continuous, and pass over smooth rounded saddles.

(b) The main chain and backstays may be separate, each secured to a saddle free to move horizontally on the top of pier.

Let  $T_1$  = tension on main chain at  $B$ .

„  $T_2$  = tension on backstay at  $B$ .

„  $\alpha_1$  = inclination to the horizontal of main chain at  $B$ .

„  $\alpha_2$  = inclination to the horizontal of backstay at  $B$ .

„  $R$  = vertical pressure on pier.

CASE A. The tensions  $T_1$  and  $T_2$  are practically equal.

Then 
$$R = T_1 (\sin \alpha_1 + \sin \alpha_2)$$

and there is a horizontal force

$$= T_1 (\cos \alpha_1 - \cos \alpha_2);$$

if  $\alpha_1 = \alpha_2$ , then 
$$R = 2T_1 \sin \alpha_1,$$

and there is no horizontal force.

CASE B. The resultant pressure on pier will always be vertical,

$$R = T_1 \sin \alpha_1 + T_2 \sin \alpha_2.$$

### 135. Stiffening Girder.

When a moving load passes over a suspension bridge the shape of the cables becomes deformed. The object of the stiffening girder is to distribute the load uniformly over the cables, so that they may not be distorted.

Fig. 281 shows a stiffening girder. The booms or chords must be designed to take tension and compression. It may be a single girder extending from tower to tower, or it may consist of two girders hinged at the centre; the latter is the better method as it counteracts the stresses due to changes of temperature.

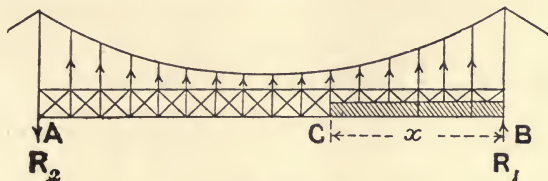


Fig. 281.

### 136. Single girder without central hinge. Uniform live load.

When the live load comes on to the bridge, the stiffening girder distributes the load uniformly to the cable, and if the load is light in comparison with the weight of the cable, the latter will keep its parabolic shape, and thus the stresses in the suspenders will be equal.

In Fig. 281,  $A$  and  $B$  are the supports, and suppose the bridge loaded over the portion  $BC = x$ , with a live load of intensity  $w$  per lineal foot.

Let  $l$  = span.

„  $P$  = pull on each suspender.

„  $p$  = uniform upward pull of the suspenders per lineal foot.

„  $R_1$  and  $R_2$  be the reactions at  $B$  and  $A$  respectively, due to the partial load.

Now on the assumption that the weight is transmitted through the suspenders,

$$pl = wx,$$

$$\text{or} \quad p = \frac{wx}{l} \dots \dots \dots (1).$$

Applying the conditions of equilibrium,

$$R_1 + R_2 + pl - wx = 0.$$

$$\therefore R_1 + R_2 = 0.$$

Taking moments round  $B$ ,

$$R_2 l + \frac{pl^2}{2} - \frac{wx^2}{2} = 0.$$

Therefore

$$R_2 = -\frac{wx}{2l}(l-x),$$

and

$$R_1 = -R_2 = \frac{wx}{2l}(l-x) \dots \dots \dots (2).$$

The reactions are therefore equal and opposite, and are a maximum when  $x = \frac{l}{2}$ ; their maximum value being  $\frac{wl}{8}$ . Thus, the maximum shearing force at the supports occurs when the live load covers half the span.

The shearing force at any section in the loaded segment distant  $x_1$  from the right support is

$$F = R_1 + px_1 - wx_1.$$

Substituting for  $R_1$  and  $p$  their values from (1) and (2),

$$F = \frac{w(l-x)}{l} \left( \frac{x}{2} - x_1 \right) \dots \dots \dots (3),$$

when

$$x_1 = \frac{x}{2}, \quad F = 0.$$

It can be similarly shown that the shearing force is zero at the middle of the unloaded segment.

Again, from (3) we see when

$$x_1 = x, \quad F = -R_1.$$

*Thus, the magnitude of the shearing force at the head of the live load is equal to that of either reaction, and the absolute maximum shearing force equal to  $\frac{wl}{8}$  occurs when the live load covers half the span.*

*Bending moment.* The bending moment at any section of the loaded segment distant  $x_1$  from the right support is

$$\begin{aligned} M &= R_1 x_1 + \frac{px_1^2}{2} - \frac{wx_1^2}{2} \\ &= \frac{w(l-x)}{2l} (xx_1 - x_1^2) \dots \dots \dots (4). \end{aligned}$$

From (4) we see that

$$M = 0, \text{ when } x_1 = 0 \text{ and when } x_1 = x.$$

*M is a max.* when  $x_1 = \frac{x}{2}$ , that is, where  $F = 0$ ,

$$\text{Max. } M = \frac{w(l-x)}{8l} x^2.$$

To find the absolute maximum as the load advances, equate

$$\frac{d}{dx} (lx^2 - x^3)$$

to zero, which gives  $x = \frac{2}{3}l$ .

Therefore absolute maximum bending moment

$$= \frac{wl^2}{54},$$

and occurs when the live load covers two-thirds of the span.

Similarly by considering the unloaded segment, the maximum bending moment occurs also at its middle section, its absolute maximum value being

$$= -\frac{wl^2}{54},$$

and occurs when the live load covers one-third of the span.

### 137. Stiffening girder hinged at the centre.

The hinge at centre provides for contraction and expansion, and thus counteracts the stresses due to changes of temperature.

As in the last case the cable is assumed to remain parabolic in shape with its vertex at the middle when the span is partially loaded, and consequently all the suspenders are subject to an equal stress. Again, owing to the hinge, there is no bending moment at the middle.

Taking the same notation as in the last case, let  $R_1$  and  $R_2$  be the reactions at the right and left supports;  $w$  the intensity of the live load; and  $p$  the uniform upward pull of the suspenders.



Fig. 282.

Let the live load, as in Fig. 282, cover a portion  $x$  of the right half span.

Then for equilibrium, we have

$$R_1 + R_2 + pl - wx = 0,$$

and taking moments about the hinge,

$$R_1 \frac{l}{2} + p \frac{l^2}{8} - \frac{wx}{2} (l - x) = 0,$$

$$R_2 \frac{l}{2} + \frac{pl^2}{8} = 0.$$

From these three equations we get

$$p = \frac{2wx^2}{l^2},$$

$$R_1 = \frac{w}{2l} (2xl - 3x^2),$$

$$R_2 = -\frac{w}{2l} x^2.$$

$R_1$  is a maximum when  $x = \frac{l}{3}$ ; and  $\max. R_1 = \frac{wl}{6}$ .

$R_2$  is a maximum when  $x = \frac{l}{2}$ ; and  $\max. R_2 = -\frac{wl}{8}$ .

The shearing force at the front of load

$$\begin{aligned} F &= R_1 + px - wx \\ &= \frac{w}{2l^2} (4x^3 - 3x^2l). \end{aligned}$$

This is a maximum when  $x = \frac{l}{2}$ , and is equal to  $\frac{wl}{8}$ .

*Maximum bending moments.*

The maximum bending moment occurs at the section where  $F = 0$ .

At any section of the loaded segment distant  $x_1$  from the right support,

$$F = R_1 + px_1 - wx_1 \dots\dots\dots (1),$$

$$M = R_1x_1 + \frac{px_1^2}{2} - \frac{wx_1^3}{2} \dots\dots\dots (2).$$

If  $F = 0$ , then from (1)

$$x_1 = \frac{R_1}{w-p} = \frac{(2xl - 3x^2)l}{2(l^2 - 2x^2)},$$

and

$$\begin{aligned} M &= \frac{1}{2} \frac{R_1^2}{w-p} \\ &= \frac{w}{8} \frac{(2xl - 3x^2)^2}{l^2 - 2x^2} \dots\dots\dots (3). \end{aligned}$$

For *max. M*, differentiating and equating to zero, we get

$$(l^2 - 2x^2)(l - 3x) + x^2(2l - 3x) = 0,$$

or

$$3x^3 - 3l^2x + l^3 = 0;$$

$$x = 0.4l$$

is an approximate solution.

Substituting in (3),

$$\text{Max. positive } M = \frac{wl^2}{53} (\text{app}).$$

For the left-hand half of the span, at a section distant  $x_2$  from the left support,

$$F = R_2 + px_2 = -\frac{wx^2}{2l} + \frac{2wx^2}{l^2} x^2 \dots\dots\dots (4),$$

$$M = R_2x_2 + \frac{px_2^2}{2} \dots\dots\dots (5).$$

From (4) we see, when  $F = 0$ ,  $x_2 = \frac{l}{4}$ ,

and

$$M = -\frac{wx^2}{16}.$$

Therefore, *max. negative bending moment*  $= -\frac{wl^2}{64}$ .

## ARCHED RIBS.

**138. Linear arch or curve of pressures.**

*Suspended and arch systems.* When a chain hangs under a distributed load of uniform intensity per unit of span, it assumes the shape of a parabola, similar to that of the bending moment curve for a beam or girder similarly loaded. There is a tension at each point of the chain, the horizontal component of which is constant. Further, if the load instead of being uniformly distributed, consists of a series of loads hanging at intervals, the chain will take up a shape corresponding to the bending moment diagram, and the bending moment at any point is proportional to the depth of the chain below the line of supports.

If, now, we suppose the chain inverted and stiffened we get an arch, and the same principles apply except that we have compression or thrust at each point of the arch instead of tension.

The curve of pressure, or linear arch, is a funicular polygon of the forces which act on the arch, and it has been shown in Art. 54, that it is the bending moment curve drawn to a definite scale for a similarly loaded horizontal beam of the same span.

If the linear arch coincides with the axis of the rib, the thrust on any normal cross section is axial, and consequently of uniform intensity.

But the arch being incapable of adjusting itself to the bending moment curve for variable loading, there is bending produced where the linear arch does not coincide with the axis of the arch, and at these sections we have bending moment and shearing stress, as well as a thrust.

**139. Bending moment and thrust in an arched rib.**

*Vertical loads.* Let  $ADCB$  (Fig. 283) be the axis of the rib, and let  $AEGB$  represent the line of pressure.

Draw a vertical line  $JDE$  cutting the axis of rib at  $D$ , and the line of pressure at  $E$ .

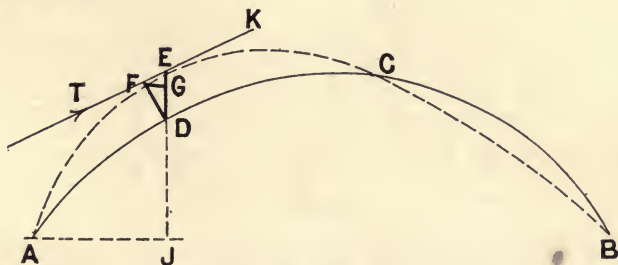


Fig. 283.

Draw  $EK$  a tangent at  $E$  to the line of pressure, and call  $T$  the thrust at  $E$ , its line of action being along the tangent  $EK$ . Draw  $DF$  perpendicular to the tangent  $EK$ , and  $FG$  perpendicular to  $DE$ . Then  $T$  at  $E$  is equivalent to a parallel force  $T$  at  $D$ , and a couple whose moment is

$$M = T \times DF.$$

The horizontal component of  $T$  is

$$H = T \cos FDE = T \frac{DF}{DE}.$$

Therefore,

$$H \times DE = T \times DF = M.$$

*Thus the moment at  $D$  is equal to the constant horizontal component of thrust multiplied by the height of the linear arch above  $D$ .*

Again, the force  $T$  at  $D$  (the centre of area of the cross section of rib) may be resolved into components parallel and perpendicular to the normal section at  $D$ ; the parallel component is the shearing stress; the perpendicular component produces a uniform compressive stress which has to be combined with the stress due to the bending moment. Thus the thrust, shear, and bending moment at any section are easily found when the funicular has been drawn.

#### 140. Arch with three hinges. Loads vertical.

The hinges are placed at the ends, and at the crown. At these three points the bending moment is zero, therefore the linear arch passes through the centre of each hinge.

In Fig. 284, let  $ACB$  be the rib, hinged at  $A$ ,  $B$ , and  $C$ , and suppose  $W$  the load acting at a distance  $x$  from  $A$ , the left support.

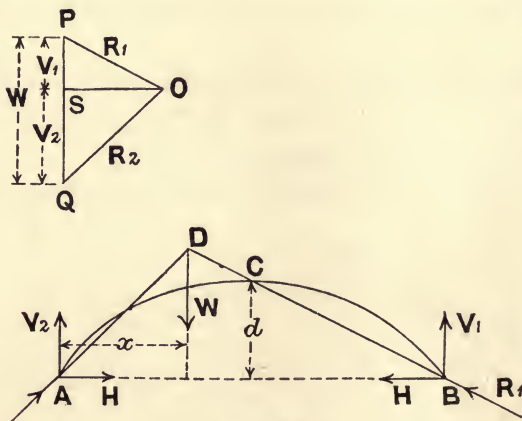


Fig. 284.

Then since there is no load on  $BC$ , the pressure at  $C$  and the reaction  $R_1$  at  $B$  must be equal and opposite, their lines of action being along

*BC*. Join *BC* and produce it to meet the line of action of *W* in *D*, join *AD*; this must be the line of action of the other reaction *R*<sub>2</sub> at *A*. These reactions may be found graphically by taking the vertical *PQ* to represent *W*, and drawing lines *PO* and *QO* parallel respectively to *R*<sub>1</sub> and *R*<sub>2</sub>. If *OS* be drawn horizontal, we get *H* the horizontal component; and *V*<sub>1</sub>, *V*<sub>2</sub>, the vertical components of *R*<sub>1</sub> and *R*<sub>2</sub>. The load being vertical the horizontal components of the two reactions must be equal.

Let *V*<sub>1</sub> and *V*<sub>2</sub> be the vertical components. Then

$$\begin{aligned} V_1 + V_2 &= W, \\ V_2 l - W(l - x) &= 0, \\ V_2 \frac{l}{2} - Hd - W\left(\frac{l}{2} - x\right) &= 0, \end{aligned}$$

where *H* is the horizontal component, and *d* the rise of arch at the crown.

From these equations

$$V_2 = \frac{W(l - x)}{l},$$

$$V_1 = \frac{Wx}{l},$$

$$H = \frac{Wx}{2d}.$$

The values of *V*<sub>1</sub> and *V*<sub>2</sub> are the same as the vertical reactions for a horizontal girder of the same span loaded in the same way.

The reactions due to a number of loads can be found by adding together the respective values of *V*<sub>1</sub>, *V*<sub>2</sub>, and *H* found for each load, or they may be found graphically. When the reactions have been obtained, the stresses in the different members may be found either analytically or graphically as in the case of an ordinary truss.

**141.** Professor Ewing gives the following method for finding the bending moments.

The linear arch must pass through the centre of each hinge. Draw the axis of rib, then draw the bending moment diagram for the given loads considered as acting on a beam of span *AB*. If this diagram passes through the third hinge, it is the true linear arch; if not, alter the scale of the bending moment diagram, drawn on the base *AB*, so as to make it pass through *C* the third hinge. This can be done by first drawing it to any scale, and then reducing all the ordinates in the ratio of the central height of axis of rib to the central ordinate of the bending moment diagram.

The linear arch having been thus drawn, the vertical distance between it and the axis of rib gives on the same scale the bending moment. The thrust *T* is found from the known form of the linear

arch and the known values of the loads. Thus the stress at any section of the rib is found. The loads may be symmetrical or unsymmetrical.

This method may also be applied to the case of a chain with hinged stiffening girder.

*Example.*

*A semicircular arched rib hinged at the crown and springing carries a uniform load of  $w$  lbs. per foot of horizontal length. Find the position and value of the maximum bending moment.*

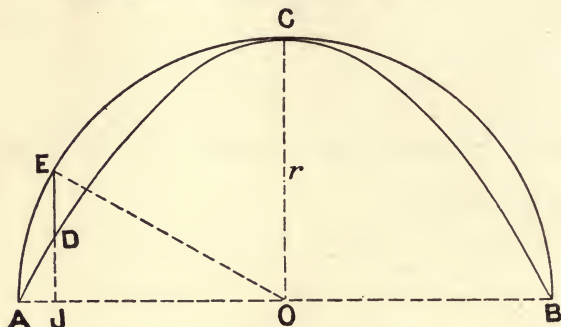


Fig. 285.

In Fig. 285, let  $AECB$  be the axis of the circular rib of radius  $r$ . The load being uniform, the line of pressures will be the parabola  $ADCB$ , passing through the hinges  $A$ ,  $C$ , and  $B$ . It has been shown that the bending moment at any point  $E$  of the rib is

$$M = H \times DE$$

where  $H$  is the horizontal thrust.

In order to find the maximum bending moment, it is first necessary to determine the maximum value of  $DE$ .

Take  $O$  the centre of circle as origin, and let us find the value of  $x(OJ)$  for which  $ED$  is a maximum.

$$\begin{aligned} \text{Now,} \quad JE &= \sqrt{r^2 - x^2}, \\ JD &= r \left( 1 - \frac{x^2}{r^2} \right). \end{aligned}$$

$$\text{Therefore,} \quad DE = \sqrt{r^2 - x^2} - \frac{1}{r} (r^2 - x^2).$$

Differentiating and equating to zero for a maximum,

$$\begin{aligned} \frac{-x}{\sqrt{r^2 - x^2}} + \frac{2x}{r} &= 0, \\ \frac{1}{r^2 - x^2} &= \frac{4}{r^2}, \\ x &= \frac{r\sqrt{3}}{2}. \end{aligned}$$

Substituting in the equation for  $DE$ , we get

$$\text{Max. } DE = \frac{r}{4}.$$

The direction of the thrust  $T$  at  $A$  is a tangent to the parabola at that point. This tangent can be at once got by producing  $OC$  to a point  $K$ , making  $CK = CO$ , and then joining  $KA$ . Now as  $OC = OA = r$ , the tangent at  $A$  makes an angle  $\theta$  with the horizontal

$$AO \text{ such that } \tan \theta = \frac{2r}{r} = 2,$$

$$\text{and, } \sin \theta = \frac{2}{\sqrt{5}}; \quad \cos \theta = \frac{1}{\sqrt{5}}.$$

$$\text{Resolving vertically, } \frac{2}{\sqrt{5}} T = wr,$$

$$\text{therefore, } T = \frac{wr \sqrt{5}}{2}.$$

Horizontal thrust,

$$H = T \cos \theta = \frac{wr \sqrt{5}}{2} \cdot \frac{1}{\sqrt{5}} = \frac{wr}{2} = \frac{W}{4},$$

where  $W$  = total weight on the arch.

Therefore, maximum bending moment

$$= H \times \text{max. } DE = \frac{wr}{2} \cdot \frac{r}{4} = \frac{wr^2}{8}.$$

### EXERCISES.

1. The span of a suspension bridge is 150 feet. The dip of the chains is 25 feet. Load 1500 lbs. per lineal foot of span. Assuming the chains to hang in parabolic curves, find the tension at the lowest point, and at the ends of each of the two chains.

2. A cable weighing 500 lbs. per horizontal foot of span is stretched between supports in the same horizontal line 1000 feet apart. If the maximum deflection is 70 feet, find the greatest and least tensions in the cable.

3. A suspension bridge consists of a central span of 240 feet and two side spans of 120 feet. The dip of central span is 20 feet, the chains of the side spans hang in a parabolic arc similar to one-half that of the central span. If the three spans are loaded with  $1\frac{1}{2}$  tons per foot of span, determine the greatest and least tensions in the chains, and the vertical and horizontal forces acting on the towers and abutments.

4. In last exercise if the saddles are fixed to the tops of the towers, and the load on the side spans is 1 ton per foot run, that on

the central span being 2 tons per foot run, find the magnitude and direction of the resultant pressure on each tower.

5. A cantilever bridge supported on piers 1, 2, 3, 4 consists of two end spans each of 200 feet, and a central span of 300 feet. The bridge is hinged in the side spans at distances of 50 feet from piers 2 and 3. Find the bending moments at pier 2 and at the middle of the central span in each of the following cases:

(a) Dead load of  $1\frac{1}{2}$  tons per foot, and live load of 2 tons per foot covering all the spans.

(b) Dead load of  $1\frac{1}{2}$  tons per foot on all spans. Live load 2 tons per foot on the spans 1—2, 3—4 only.

(c) Same dead load on all spans. Same live load on the central span only.

6. A suspension bridge is formed of two cables of uniform section, span 120 feet, dip 12 feet. Width of bridge 10 feet. Load 150 lbs. per square foot. Find the maximum tension on the cables, and their cross sectional area if the working stress is 5 tons per square inch.

7. A cantilever bridge supported on piers 1, 2, 3, 4 consists of two end spans each 100 feet and a central span 260 feet. The hinges are in the central span at 56 feet from piers 2 and 3. Find the bending moments at pier 2, and at the middle of the central span, taking a dead load on all the spans of  $1\frac{1}{2}$  tons per foot run, and a live load of 2 tons per foot run distributed as follows:

(a) Live load covering all spans.

(b) Live load covering the centre portion between the two hinges.

(c) Live load covering the two portions from hinges to the ends 1 and 4, leaving the central portion between hinges uncovered.

8. A footpath 10 feet wide is to be carried over a river 80 feet wide by two cables of uniform section. Dip at the centre 10 feet. Load 160 lbs. per square foot. Determine (a) the maximum pull on the cables; (b) the necessary cross-sectional area; if the working stress in the cable is not to exceed  $4\frac{1}{2}$  tons per square inch, and if the material of the cable weighs 480 lbs. per cubic foot.

## CHAPTER XIV.

### TORSION.

#### 142. Theory of torsion or twisting.

When a cylindrical bar or shaft of uniform section is fixed at one end, and twisted by a single couple at the free end in a plane perpendicular to the axis of the bar (Fig. 286); or what is the same, if a pair of equal and opposite couples are applied to the ends, the axis of

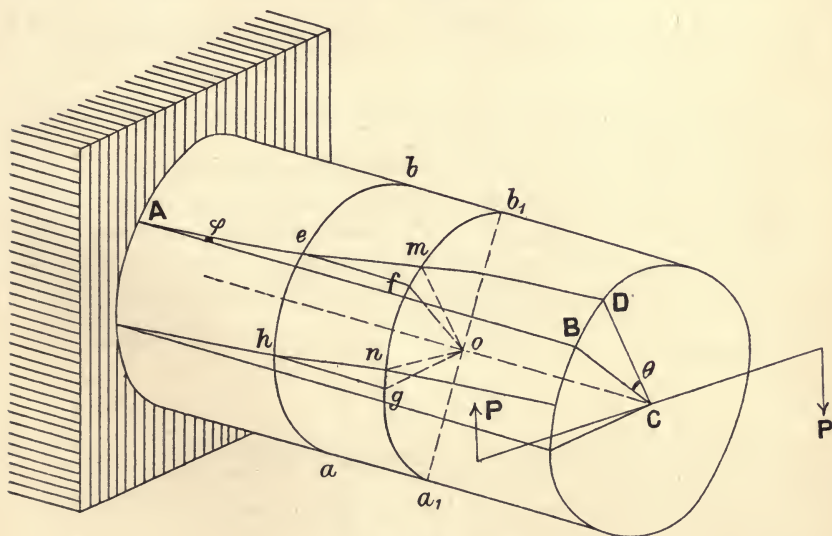


Fig. 286.

the couples coinciding with the axis of the bar; the effect of the couples is to turn one transverse section of the bar relatively to another through a small angle, and to cause fibres originally straight and parallel to the axis, such as  $AB$ , to become changed into helices, as  $AD$ , inclined at a constant angle  $\phi$  to the axis. A small square  $efgh$  drawn on the surface of the bar becomes distorted into a rhombus  $emnh$  corresponding

exactly to the deformation produced by shearing stress. At any transverse section the *resistance to torsion is the shearing stress exerted at the section*, which is equivalent to tensile and compressive stresses, of equal intensity to the shearing stress, acting along the diagonals  $hm$ ,  $en$ , inclined at  $45^\circ$  to the axis of the bar. Thus, the lines of principal stress are helices inclined at  $45^\circ$  to the axis of bar (Fig. 287).

The strain at any point in a cross section is evidently proportional to the distance of the point from the axis; consequently within the elastic limits, the *shearing stress* which is at right angles to the radius drawn to the point, has an intensity  $q$  proportional to that radius.

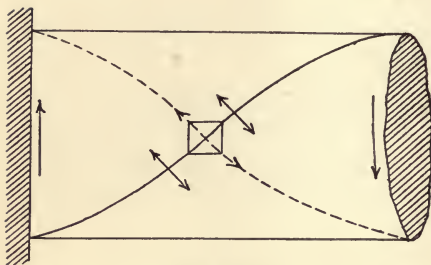


Fig. 287.

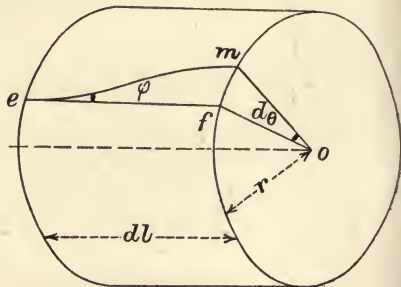


Fig. 288.

Let Fig. 288 represent two cross sections of a bar of radius  $r$ , taken at a very small distance  $dl$  from one another.

Let  $\phi$  be the angle which the helix  $em$  makes with the line  $ef$  originally parallel to the axis: then  $\phi$  is the *angle of shear*, and it is proportional to  $r$ . The radius  $of$  is turned through a small angle  $d\theta$ , called the *angle of twist*, or *angle of torsion*, which is proportional to  $dl$ .

Now the distance  $fm$  through which one section turns relatively to the other is expressed by  $\phi dl$ , or  $rd\theta$ .

Therefore

$$\phi dl = rd\theta,$$

or

$$\phi = r \frac{d\theta}{dl}.$$

The corresponding intensity of shearing stress is

$$q = C\phi = Cr \frac{d\theta}{dl} \dots\dots\dots(1),$$

where  $C$  is the modulus of rigidity.

$\frac{d\theta}{dl}$ , the angle twist per unit of length, is constant.

Therefore  $q$  varies as  $r$ .

### 143. Maximum intensity of shearing stress in circular shafts to a given twisting moment.

Let  $M_T$  = the twisting moment, or moment of the twisting couple.

Let  $r_1$  = the radius of the shaft.

„  $r$  = radius of a concentric cylindrical surface in the interior of the shaft.

„  $q_1$  = maximum intensity of shearing stress, which occurs at the surface, that is, at a distance  $r_1$  from the axis.

„  $q$  = intensity of shearing stress at distance  $r$  from the axis.

Consider a thin ring of cross section of radius  $r$ , and thickness  $dr$  (Fig. 289).

The shearing stress on this small area is

$$q2\pi r dr.$$

The moment of this shearing stress round the axis is

$$q2\pi r^2 dr.$$

The total moment of the shearing stress distributed over the cross section is

$$\int q2\pi r^2 dr = \frac{2\pi q_1}{r_1} \int r^3 dr,$$

since

$$\frac{q}{r} = \frac{q_1}{r_1}.$$

For equilibrium, this moment must be equal and opposite to the twisting moment.

Therefore 
$$M_T = \frac{2\pi q_1}{r_1} \int r^3 dr \dots \dots \dots (2).$$

For a *solid shaft*, integrating between the limits  $r_1$  and 0, we get

$$M_T = \frac{2\pi q_1}{r_1} \int_0^{r_1} r^3 dr = \frac{\pi q_1 r_1^3}{2} = \frac{q_1}{r_1} J \dots \dots \dots (3),$$

where  $J$  is the *polar moment of inertia* of the section, which is equal to twice the moment of inertia about a diameter.

Or, maximum intensity of shearing stress

$$q_1 = \frac{2M_T}{\pi r_1^3} = \frac{16M_T}{\pi d_1^3} \dots \dots \dots (4),$$

if  $d_1$  is the diameter of the shaft.

For a *hollow shaft*, of external radius  $r_1$ , and internal radius  $r_2$ , we have from equation (2)

$$\begin{aligned} M_T &= \frac{2\pi q_1}{r_1} \int_{r_2}^{r_1} r^3 dr \\ &= \frac{\pi q_1 (r_1^4 - r_2^4)}{2r_1} = \frac{\pi q_1}{16} \cdot \frac{(d_1^4 - d_2^4)}{d_1} \dots \dots \dots (5), \\ &= \frac{q_1}{r_1} J \end{aligned}$$

where  $d_1$  and  $d_2$  are the external and internal diameters, and  $J$  is the polar moment of inertia of the section.

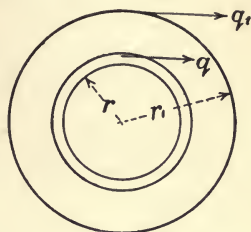


Fig. 289.

The maximum intensity of shearing stress

$$q_1 = \frac{2M_T r_1}{\pi (r_1^4 - r_2^4)} = \frac{16M_T d_1}{\pi (d_1^4 - d_2^4)} \dots\dots\dots(6).$$

These equations are true only so long as the stress does not exceed the elastic limit.

The *working values* of  $q_1$  for steady motion may be taken as:

For cast iron	3600 lbs. per sq. inch.
For wrought iron	9000       ,,       ,,
For steel	13500     ,,       ,,

The above results apply only to circular cross sections in which the ratio of  $q$  to  $r$  is constant.

**144. Angle of torsion for a shaft of uniform circular cross section. Experimental determination of  $C$ , the modulus of rigidity.**

Let  $\theta$  = the total angle of torsion or twist, that is, the angle *in circular measure* through which the section in the plane of one twisting couple is turned relatively to the section in the plane of the other.

„  $l$  = length of the shaft between the two couples.

Then  $\frac{\theta}{l}$  is the angle of torsion *per unit of length*, and from equation (1)

$$\frac{\theta}{l} = \frac{q}{Cr} = \frac{q_1}{Cr_1};$$

but, by equation (4), *within the elastic limits*,

$$\frac{q_1}{r_1} = \frac{2M_T}{\pi r_1^4}.$$

Therefore, for a *solid* shaft,

$$\frac{\theta}{l} = \frac{2M_T}{C\pi r_1^4} = \frac{32M_T}{C\pi d_1^4} = \frac{M_T}{CJ},$$

$$\text{or,} \quad \theta = \frac{32M_T l}{C\pi d_1^4} = \frac{10 \cdot 2M_T l}{Cd_1^4} \dots\dots\dots(7).$$

Similarly, for a *hollow* shaft,

$$\frac{\theta}{l} = \frac{2M_T}{C\pi (r_1^4 - r_2^4)} = \frac{32M_T}{C\pi (d_1^4 - d_2^4)} = \frac{M_T}{CJ},$$

$$\text{or,} \quad \theta = \frac{32M_T l}{C\pi (d_1^4 - d_2^4)} = \frac{10 \cdot 2M_T l}{C(d_1^4 - d_2^4)} \dots\dots\dots(8).$$

The quantity  $\frac{M_T}{\left(\frac{\theta}{l}\right)} = CJ$  measures the torsional rigidity of the shaft.

These latter equations are used for the determination of  $C$ .

The usual way of determining the Modulus of Rigidity is by

experiments on torsion, in which the stresses are within the elastic limit. The following is a very simple method for thin rods. The rod is fixed at one end; on the free end is clamped a lever arm and pointer. Before starting the experiment, the fixed end, which is capable of adjustment by means of a slotted arc, is moved till the lever arm is truly horizontal and the pointer marks zero on a fixed graduated scale. The end of the lever arm is now weighted, and the pointer measures on the scale the angle of torsion in degrees. The rod rests freely on supports.

The following is an experiment made on a wrought iron rod 12 inches long,  $\frac{1}{4}$  inch diameter, fixed at one end and carrying a lever arm 10 inches long at the free end. The arm was first set horizontal, and the pointer marked zero. A weight of 6 lbs. was then hung at the end of the lever arm, and the pointer measured  $9^\circ$  on its scale.

Then the twisting moment =  $10 \times 6 = 60$  inch-lbs.

Angle of torsion in circular measure =  $\frac{\pi \times 9^\circ}{180}$ ,

$l = 12$  inches;  $d_1 = \frac{1}{4}$  inch.

From equation (7),  $\theta = \frac{32M_T l}{C\pi d_1^4}$ .

Therefore  $C = \frac{32M_T l}{\pi d_1^4 \theta}$   
 $= \frac{32 \times 60 \times 12 \times 7 \times 7 \times 256 \times 180}{22 \times 22 \times 9 \times 2240}$  tons per sq. inch  
 $= 5331$  tons per sq. inch.

**145. Relation between twisting moment and horse-power transmitted. Diameter of a round shaft to transmit a given horse-power.**

Let  $F$  = the force of the twisting couple, in pounds, either constant, or the mean value if variable.

„  $R$  = the length of the lever arm of the twisting couple in feet.

Then the mean twisting moment  $M_T = F \cdot R$  foot-lbs.

Let  $HP$  = number of horse-power transmitted; one horse-power being 33000 foot-lbs. of work done per minute.

„  $N$  = number of revolutions per minute.

The work done per minute =  $F \times 2\pi RN$  foot-lbs.

$= M_T \times 2\pi N$  foot-lbs.

Again, the work done per minute = 33000  $HP$  foot-lbs.

Therefore,  $M_T = \frac{33000 HP}{2\pi N}$ ,

or,  $M_T$  (inch-tons) =  $\frac{12 \times 33000 HP}{2\pi N} = 63030 \frac{HP}{N}$  (inch-tons).

By equation (3), 
$$M_T = \frac{\pi q_1 d_1^3}{16}.$$

Hence, 
$$\frac{\pi q_1 d_1^3}{16} = 63030 \frac{HP}{N}.$$

Therefore 
$$d_1 = 68.5 \sqrt[3]{\frac{HP}{Nq_1}}.$$

Assuming the safe values of  $q_1$  to be for steady motion,

Cast iron 3600 lbs. per sq. inch.

Wrought iron 9000 „ „

Steel 13500 „ „

The diameter in inches for a round shaft in terms of the horsepower is for

Cast iron, 
$$d_1 = 4.5 \sqrt[3]{\frac{HP}{N}}.$$

Wrought iron, 
$$d_1 = 3.3 \sqrt[3]{\frac{HP}{N}}.$$

Steel, 
$$d_1 = 2.9 \sqrt[3]{\frac{HP}{N}}.$$

The twisting moment is here assumed to remain constant at its mean value. In practice, however, the twisting moment in many cases varies, to allow for which it is usual to take the maximum twisting moment as from 1.3 to 1.5 times the mean twisting moment, thus slightly increasing the values of  $d_1$  as found above.

*Work done in twisting a round shaft.*

When a twisting moment  $M_T$  is gradually increased from 0 to  $M_T$ , the angle of torsion increasing from 0 to  $\theta$ ,

$$\text{The work done} = \frac{1}{2} M_T \theta.$$

If  $l$  is the length of the shaft, we know from equation (1)

$$\frac{\theta}{l} = \frac{q_1}{Cr_1},$$

and from equation (3) 
$$M_T = \frac{\pi q_1 r_1^3}{2}.$$

Therefore, 
$$\begin{aligned} \text{The work done} &= \frac{q_1^2 \pi r_1^2 l}{4C} \\ &= \frac{q_1^2}{4C} \cdot \text{vol.} \end{aligned}$$

and, 
$$\text{The work done per unit of volume} = \frac{q_1^2}{4C}.$$

#### 146. Circular shafts subjected to twisting and bending.

In this case the shaft is acted on by a bending load, such as a heavy wheel or pulley fixed on it, in addition to a pair of twisting couples.

Let  $M_B$  and  $M_T$  be the values of the given bending moment and twisting moment respectively, occurring simultaneously on a circular cross section of a shaft of radius  $r_1$ .

„  $f$  = maximum intensity of the normal longitudinal stress, tensile or compressive, due to the bending moment  $M_B$ .

„  $q$  = maximum intensity of shearing stress at the circumference of section due to the twisting moment  $M_T$ .

Then

$$f = \frac{4}{\pi r_1^3} M_B,$$

and

$$q = \frac{2}{\pi r_1^3} M_T;$$

$f$  and  $q$  act in planes at right angles to one another.

To find the principal stress we must combine  $f$  and  $q$  as in Art. 42, where it was shown that the maximum principal stress

$$f_1 = \frac{f}{2} + \sqrt{\frac{f^2}{4} + q^2}.$$

Substituting for  $f$  and  $q$  in this equation their values as above, in terms of  $M_B$  and  $M_T$ , we see that the shaft is subjected to the maximum compressive or tensile stress

$$f_1 = \frac{2}{\pi r_1^3} \{M_B + \sqrt{M_B^2 + M_T^2}\} \dots\dots\dots (9)$$

for a solid shaft, or

$$f_1 = \frac{2r_1}{\pi (r_1^4 - r_2^4)} \{M_B + \sqrt{M_B^2 + M_T^2}\} \dots\dots\dots (10)$$

for a hollow shaft.

Thus the maximum principal stress in a shaft, due to the combined effect of  $M_B$  and  $M_T$ , has the same value as the stress due to the bending moment acting alone without twisting

$$= \frac{1}{2} (M_B + \sqrt{M_B^2 + M_T^2}),$$

called the *equivalent bending moment*.

Or, the maximum principal stress is numerically equal to the greatest shearing stress which would be produced by a twisting moment acting alone

$$= M_B + \sqrt{M_B^2 + M_T^2},$$

called the *equivalent twisting moment*.

If the working tensile or compressive stress ( $f_1$ ) is given, the corresponding value of  $r_1$  may be got from equation (9).

The maximum shearing stress, due to twisting and bending, by

$$\begin{aligned} \text{Art. 42, is} &= \sqrt{\frac{f^2}{4} + q^2} \\ &= \frac{2 \sqrt{M_B^2 + M_T^2}}{\pi r_1^3}. \end{aligned}$$

In an ordinary crank shaft (Fig. 290), let  $F$  be the force applied to the pin  $A$ , at right angles to the crank.

Then  $M_B = F \cdot BC$ ,

and  $M_T = F \cdot AB$ .

Hence

$$\begin{aligned} f_1 &= \frac{2F}{\pi r_1^3} \{BC + \sqrt{BC^2 + AB^2}\} \\ &= \frac{2F}{\pi r_1^3} \{BC + AC\}. \end{aligned}$$

The maximum shear stress

$$= \frac{2F}{\pi r_1^3} \sqrt{BC^2 + AB^2} = \frac{2F \cdot AC}{\pi r_1^3}.$$

*Examples.*

1. Find the principal stress in a shaft 10 inches diameter, 20 feet long between the bearings, weighing 0.12 ton per foot run. The shaft carries a wheel weighing 3 tons at 4 feet from the right-hand bearing, and transmits 800 horse-power at 100 revolutions per minute.

The reaction at the right-hand bearing is

$$R_1 = \frac{3 \times 16}{20} + \frac{20}{2} \times 0.12 = 3.6 \text{ tons.}$$

Let  $x$  be the distance from right-hand bearing at which the bending moment  $M_B$  is a max., that is, where the shearing force  $F = 0$ .

Now  $F = 0$ , at  $x = 5$  feet.

$$\begin{aligned} \text{Hence } M_B &= 3.6 \times 5 - 3 \times 1 - 0.12 \times \frac{2.5}{2} \\ &= 13.50 \text{ foot-tons} = 162 \text{ inch-tons.} \end{aligned}$$

The twisting moment is constant for all sections, and

$$\begin{aligned} M_T &= \frac{33000 \times HP}{2\pi N} = \frac{33000 \times 800 \times 7}{2 \times 22 \times 100} \\ &= 42000 \text{ foot-lbs.} \\ &= 225 \text{ inch-tons.} \end{aligned}$$

The greater principal stress

$$\begin{aligned} f_1 &= \frac{2}{\pi r_1^3} \{M_B + \sqrt{M_B^2 + M_T^2}\} \\ &= \frac{2 \times 7}{22 \times 125} \{162 + \sqrt{162^2 + 225^2}\} \\ &= \frac{7}{1375} \times 439 = 2.23 \text{ tons per sq. inch.} \end{aligned}$$

#### 147. Torsion of shafts not circular in cross section.

St Venant has investigated the stress produced by torsion in shafts of elliptic, square, and other cross sections. In a circular section the

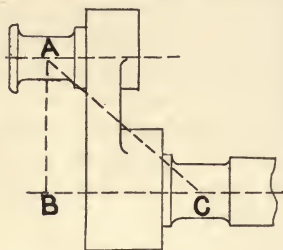


Fig. 290.

stress is the same at all points equally distant from the centre, varying directly as the radius  $r$ ; and sections primitively plane and normal remain plain and normal during twisting. St Venant has shown that in non-circular sections the stress is greatest at those points of the boundary of the section which are nearest to the centre of area. The following are the values for  $M_T$  and  $\theta$  (in circular measure for different sections). Those for a circle are restated for comparison.

*Section a circle of diameter  $d$ ,*

$$M_T = \frac{\pi}{16} q_1 d^3,$$

$$\theta = \frac{10 \cdot 2 M_T l}{C d^4}.$$

*Section an ellipse. Axis major =  $a$ . Axis minor =  $b$ .*

$$M_T = \frac{\pi}{16} q_1 a b^2.$$

The maximum stress occurs at the extremities of the smaller axis  $b$ .

$$\theta = \frac{5 \cdot 1 M_T l (a^2 + b^2)}{C a^3 b^3}.$$

*Section a square of side  $b$ ,*

$$M_T = 0 \cdot 208 q_1 b^2.$$

Maximum stress occurs at the middle points of each side.

$$\theta = \frac{7 \cdot 1 M_T l}{C b^4}.$$

*Section a rectangle; sides =  $B$  and  $b$ ;  $B > b$ .*

$$M_T = n q_1 B b^2.$$

Maximum stress occurs at the middle of the longer side.

$$\theta = \frac{3 \cdot 5 M_T l (B^2 + b^2)}{C B^3 b^3}.$$

$n$  in the above equation for a rectangle is a numerical coefficient the value of which varies for different ratios of  $B$  to  $b$ .

$\frac{B}{b}$	$n$	$\frac{B}{b}$	$n$
1	0·208	3·5	0·275
1·5	0·231	4	0·282
2	0·246	5	0·292
2·5	0·258	10	0·312
3	0·267		

### 148. Cylindrical spiral springs.

Fig. 291 represents a spiral spring of length  $l$ , loaded with a weight  $W$  in the direction of its axis.

Let  $R$  be the radius of the coil measured to the centre of the wire from the axis of the spring and  $r$  the radius of the wire.

The strain is practically pure torsion, for although there is some bending, it is comparatively small. Taking any normal cross section of the wire, we have there a shearing force  $W$  and a twisting torque

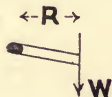
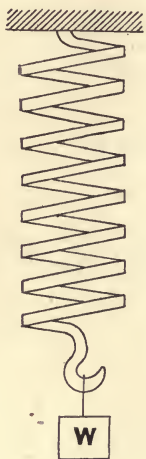


Fig. 291.

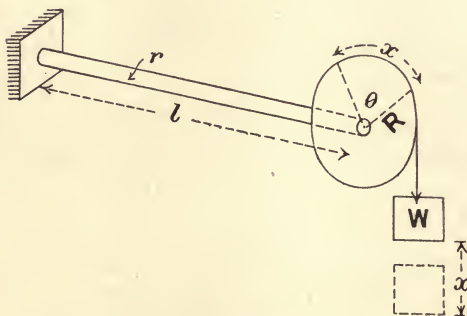


Fig. 292.

equal to  $WR$ . The effect of the shearing force  $W$  may be neglected in comparison with that of the moment  $WR$ , more especially when the spring is closely coiled, and the radius  $R$  of the coil is large as compared with the radius  $r$  of the wire. Thus the spring may be assumed as being subjected to torsion at every section.

The twisting moment =  $WR$  is the same for every section of the wire.

Hence, the maximum shearing stress

$$q_1 = \frac{2WR}{\pi r^3} = \frac{C\theta r}{l},$$

where  $\theta$  is the total angle of torsion for the length  $l$ .

If  $n$  be the number of coils in the spring, the length

$$l = 2\pi R \cdot n.$$

The *elongation* of the spring by the load  $W$  (Fig. 292)

$$x = R\theta = \frac{2WR^2l}{C\pi r^4} = \frac{q_1 Rl}{Cr}.$$

The work done in stretching the spring, or the *energy stored in the spring*

$$= \frac{WR}{2} \theta = \frac{W^2 R^2 l}{C\pi r^4} = \frac{q_1^2}{4C} \pi r^2 l.$$

The work done per unit of volume of the wire  $= \frac{q_1^2}{4C}$ .

Mr Hartnell gives the following values found from experiment for the maximum safe stress for steel wire:

Wire  $\frac{1}{4}$  inch diameter 70000 lbs. per sq. inch.

„  $\frac{3}{8}$  „ „ 60000 „ „ „

„  $\frac{1}{2}$  „ „ 50000 „ „ „

2. A spiral spring has 20 coils, the diameter of the coil is 4 ins., and the diameter of the steel rod of which the spring is made is  $\frac{1}{4}$  inch. Find the weight  $W$  acting along the axis of the coil which will produce an extension of 3 ins.  $C = 5000$  tons per sq. inch.

If  $n$  = number of coils in the spring, the length

$$l = 2\pi Rn.$$

$$\text{The extension} = \frac{q_1 Rl}{Cr}.$$

Hence

$$3 = \frac{q_1 \times 2 \times \frac{2.2}{7} \times 4 \times 20}{5000 \times \frac{1}{8}}.$$

Therefore

$$q_1 = \frac{3 \times 500 \times 7}{2 \times 22 \times 4 \times 8 \times 2} = 3.7 \text{ tons per sq. inch.}$$

Now

$$\text{Resilience} = \frac{q_1^2}{4C} V.$$

Hence

$$\begin{aligned} \frac{W}{2} \times 3 &= \frac{3.7 \times 3.7}{4 \times 5000} \times \frac{22}{7} \times 4 \times \frac{22}{4 \times 7} \times \frac{1}{16} \times 20 \times 2240 \\ &= 18.9 \text{ inch-lbs.} \end{aligned}$$

Therefore  $W = 12.9$  lbs.

3. A truck weighing 30 cwt. travels at 6 miles an hour. Find how many spiral springs each of 20 coils are necessary to store the energy of motion. The diameter of the coil in each spring is 10 ins., and the diameter of the steel rod 1 inch. The compression of the springs may be 10 ins., but not more. Take  $C = 5000$  tons per sq. inch.

First find the energy of motion due to velocity.

$$\text{Velocity} = \frac{6 \times 5280}{60 \times 60} = 8.8 \text{ ft. per sec.}$$

$$\begin{aligned}\text{Energy to be absorbed} &= \frac{Wv^2}{2g} = \frac{3}{2} \times \frac{8.8 \times 8.8}{2 \times 32.2} = 1.81 \text{ foot-tons} \\ &= 21.72 \text{ inch-tons.}\end{aligned}$$

$$\text{The compression} = \frac{q_1 Rl}{Cr}.$$

$$\text{Therefore} \quad 10 \text{ ins.} = \frac{q_1 \times 5 \times \pi \times 10 \times 20}{5000 \times \frac{1}{2}},$$

$$\text{or} \quad q_1 = \frac{25}{\pi} = 7.97 \text{ tons per sq. inch.}$$

$$\begin{aligned}\text{The resilience of one spring} &= \frac{q_1^2}{4C} \cdot V \\ &= \frac{7.97 \times 7.97}{4 \times 5000} \times \frac{\pi}{4} \times \pi \times 10 \times 20 \\ &= \frac{6.25}{4} = 1.56 \text{ inch-tons.}\end{aligned}$$

The number of springs necessary is therefore

$$= \frac{21.72}{1.56} = 14.$$

To find the weight of these 14 springs: 1 cubic inch of steel weighs 0.28 lbs.

$$\text{Volume} = 14 \times \pi r^2 \times 2\pi Rn.$$

$$\begin{aligned}\text{Therefore} \quad \text{the weight} &= \text{Vol.} \times 0.28 \\ &= 1936 \text{ lbs.}\end{aligned}$$

### EXERCISES.

1. A shaft 10 feet long,  $2\frac{1}{2}$  ins. diameter, is fixed at one end, and at the free end is twisted by a force of 500 lbs. acting at a radius of 3 feet. Find the angle of torsion, and the displacement of the point of application of the force. Take  $C = 5200$  tons per sq. inch.

*Ans.* 2.77 degrees. 1.75 inches.

2. A shaft 3 inches diameter, 15 feet long, transmits 20 horsepower at 100 revolutions per minute. Find the angle of torsion, and the maximum intensity of shearing stress. Assume  $C = 5200$  tons per sq. inch.

*Ans.* 1.37 degrees. 2380 lbs. per sq. inch.

3. A shaft 20 feet long and  $2\frac{1}{2}$  inches diameter is subjected to a twisting moment of 16000 inch-lbs., and is loaded at the centre by a pulley weighing 200 lbs. Find the equivalent twisting moment and the maximum stress.

*Ans.* 32000 inch-lbs. 10445 lbs. per sq. inch.

4. Find the diameter of a shaft required to transmit 200 horse-power at 100 revolutions per minute, the limiting intensity of shearing stress being 9000 lbs. per sq. inch.

*Ans.* 4.14 ins.

5. A steel shaft is supported in bearings 12 feet apart, and at 3 feet from one bearing carries a wheel weighing 4 tons. The shaft transmits 200 horse-power at 80 revolutions per minute. Neglecting the weight of shaft, determine its diameter for a limiting intensity of tensile stress of 6 tons per sq. inch.

*Ans.* 6 ins. (nearly).

6. Calculate the greatest intensity of tensile stress in a shaft 12 ins. diameter, 20 feet long between the bearings, weighing 480 lbs. per cubic foot. The shaft carries a wheel weighing 5 tons at 4 feet from one bearing, and transmits 1000 horse-power at 100 revolutions per minute.

*Ans.* 1.88 tons per sq. inch.

7. A circular shaft is twisted by a force of 500 lbs. applied tangentially at the circumference of a pulley, 4 feet diameter, keyed on the shaft at one end, and tending to turn it against a resistance of 1000 lbs. applied at the end of a crank 1 ft. long, keyed on the other end of the shaft, which is supported in bearings placed close to the wheel and crank respectively. Determine the diameter of the shaft, assuming the working intensity of shearing stress to be 4 tons per sq. inch.

*Ans.* 2 inches (app.).

8. If, in exercise 7, the distance between the wheel and crank is 10 feet, determine the angle of torsion, assuming  $C = 11,200,000$  lbs. per sq. inch.

*Ans.* 5.5 degrees.

9. Determine the maximum stress produced at the circumference of a circular steel engine-shaft 5 inches diameter, 10 feet long between centres of journals. The shaft carries midway between the journals a fly-wheel 12 feet diameter, weighing 4 tons, transmitting 180 horse-power at 100 revolutions per minute by means of a belt driving horizontally from the lowest point of its circumference. Take the weight of the shaft as 480 lbs. per cubic foot.

*Ans.* 10.7 tons per sq. inch.

10. Compare (1) the resistance to a steady twist, (2) the angles of torsion for the same maximum stress, (3) the resilience, of two

shafts of the same length, weight, and material, one of which is solid, and the other hollow with internal radius half the external.

$$\text{Ans.} \quad (1) \frac{2\sqrt{3}}{5}; \quad (2) \frac{2}{\sqrt{3}}; \quad (3) \frac{4}{5}.$$

11. If the amount of twist in a solid shaft is limited to  $1^\circ$  for each 10 feet in length, determine the diameter if the shaft is subjected to a twisting moment of 70 inch-tons. Take  $C = 12,000,000$  lbs. per square inch.

$$\text{Ans.} \quad 5.5 \text{ inches.}$$

12. Find the extension which a weight of 50 lbs. will produce in a spiral spring of 30 coils, if the mean diameter of the coil is  $2\frac{1}{2}$  inches, and the diameter of the wire  $\frac{1}{4}$  inch. Assume  $C = 5000$  tons per sq. inch.

$$\text{Ans.} \quad 4.3 \text{ inches.}$$

13. A rod 12 inches long and  $\frac{1}{4}$  inch diameter, fixed at one end, is twisted by a force of 6 lbs. acting at the end of a lever arm 10 inches long, which is keyed to the rod at the free end. Find the angle of torsion, and the maximum stress produced in the rod. Take  $C = 5000$  tons per sq. inch.

$$\text{Ans.} \quad 7\frac{1}{2} \text{ degrees.} \quad 8.7 \text{ tons per sq. inch.}$$

14. Show that if  $r_1$  is the outside radius of a solid or a hollow shaft, and  $q_1$  the maximum intensity of shearing stress due to a twisting moment  $M_T$ ,

$$\frac{q_1}{r_1} = \frac{M_T}{J},$$

where  $J$  is the polar moment of inertia of the section, which is equal to twice the moment of inertia about a diameter.

15. A cylindrical shaft of wrought iron transmits 100 horsepower at 60 revolutions per minute. It is supported in bearings 8 feet apart, and at 2 feet from one bearing carries a wheel weighing 3 tons. Determine the diameter such that the maximum intensity of stress shall be 4 tons per square inch.

$$\text{Ans.} \quad 5\frac{1}{2} \text{ ins. (app.).}$$

16. The external diameter of a hollow steel shaft is 10 ins., the internal diameter 8 ins. Find the twisting moment it can transmit with a working stress of 4 tons per sq. inch.

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